ENDPERIODIC MAPS VIA PSEUDO-ANOSOV FLOWS

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Abstract. We show that every atoroidal endperiodic map of an infinite-type surface can be obtained from a depth one foliation in a fibered hyperbolic 3-manifold, reversing a well-known construction of Thurston. This can be done almost-transversely to the canonical suspension flow, and as a consequence we recover the Handel–Miller laminations of such a map directly from the fibered structure. We also generalize from the finite-genus case the relation between topological entropy, growth rates of periodic points, and growth rates of intersection numbers of curves. Fixing the manifold and varying the depth one foliations, we obtain a description of the Cantwell–Conlon foliation cones and a proof that the entropy function on these cones is continuous and convex.

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1. Introduction

Let L be an infinite-type surface with finitely many ends and without boundary. A homeomorphism $g\colon L \to L$ is **endperiodic** if each end of L is either attracting or repelling under a power of g, and **atoroidal** if it fixes no finite essential multicurve up to isotopy (see Section 2.1 for precise definitions).

Such maps appear naturally in Thurston's work on fibered compact 3-manifolds: A fiber S can be "spun" around a sufficiently nice surface Σ yielding a foliation in which Σ is a compact leaf and its complement is fibered by parallel copies of a noncompact surface L, so that the monodromy of this fibering is an endperiodic map of L, which must be atoroidal when M is hyperbolic.

In this paper we reverse this process, obtaining any atoroidal endperiodic map from some fibration by a spinning operation in a suitable hyperbolic fibered 3-manifold. More importantly, the resulting foliation is transverse to a pseudo-Anosov flow (as in Fried [Fri79]), and the stable and unstable foliations of this flow induce a similar structure on L (see Theorem A

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and Theorem B). We call the return map of such a construction a *spun pseudo-Anosov (spA) map*.

This construction has several consequences:

- We recover, directly from the pseudo-Anosov structure, the dynamical laminations of Handel–Miller. See Theorem F and [Fen92, Fen97, CC99, CCF19].
- We identify dynamical growth rates of the spun pseudo-Anosov map: the spA map minimizes the exponential growth rate of periodic points among all homotopic endperiodic maps. Further, this rate is equal to the exponential growth rate of intersection numbers of curves under iteration and its log is the topological entropy (suitably defined) of the spA map. See Theorem C.
- The compactified mapping torus of the endperiodic map is a manifold N with boundary, which can admit a variety of depth one foliations whose compact leaves are ∂N. These foliations are parameterized by the *foliation cones* of Cantwell–Conlon, which are analogous to the cones on fibered faces of Thurston's norm. This analogy can be made explicit by the spinning construction, and we show that the foliation cones are exactly the pullbacks of Thurston fibered cones by the inclusion of N into a certain fibered manifold M (Theorem D). From this we can show that topological entropy defines a continuous, convex function on each foliation cone (Theorem E).

1.1. **Motivation from fibered face theory.** Let M be a closed hyperbolic manifold that fibers over the circle. Each fiber S in a fibration of M over the circle determines (and is determined by) its homology class, which is Poincaré dual to an integral element of $H^1(M;\mathbb{R})$ called a *fibered class*. The various fibrations of M are organized by the *Thurston norm*. This is a norm x on the vector space $H^1(M;\mathbb{R})$ whose unit ball $B_x = B_x(M)$ is a finite sided polyhedron having the following property: there is a set of open cones over top-dimensional faces of B_x, called *fibered cones*, so that each primitive integral point in the fibered cone $\mathcal{C}$ is a fibered class, and hence corresponds to a fiber in some fibration of M over the circle. Conversely, each fibered class lies interior to a fibered cone [Thu86].

Fried later reinterpreted Thurston's theory in terms of the **pseudo-Anosov suspension flow** $\varphi = \varphi_{\mathcal{C}}$ that is canonically associated to the fibered cone $\mathcal{C}$, up to isotopy and reparametrization [Fri79]. Each fiber surface S dual to a class in $\mathcal{C}$ is isotopic to a *cross section* of the flow φ, i.e. it is transverse and meets each orbit infinitely often, and the associated first return map to S is the pseudo-Anosov representative of its monodromy.

Since any fibered class α in $\mathcal{C}$ determines a pseudo-Anosov first return map $f\colon S \to S$ on the associated cross section S of φ, we can assign to α the logarithm of the stretch factor of f, i.e. its *entropy*, and denote this quantity by $\mathrm{ent}(\alpha)$. Fried proves that this assignment extends to a function $\mathrm{ent}\colon \mathcal{C} \to (0,\infty)$ that is continuous, convex, and blows up at the boundary of $\mathcal{C}$ [Fri82, Theorem E]. McMullen extends Fried's result by showing that ent is additionally real analytic and strictly convex [McM00, Corollary 5.4]. Understanding properties of the entropy function has since figured prominently into both the study of fibered cones as well as pseudo-Anosov stretch factors [FLM11, KKT13, Hir10, LM13, McM15].

At the boundary. The structure discussed above applies only to fiber surfaces, i.e. those representing classes in the interior of the fibered cone. However, in the article where his norm is introduced, Thurston explains the geometric significance of a taut surface Σ representing a class in the *boundary* of the fibered cone $\mathcal{C}$. Recall that Σ is **taut** if it has no nullhomologous components and it is norm minimizing, that is $x([\Sigma]) = -\chi(S)$. According to [Thu86, Remark, p. 121], there are foliations $\mathcal{F}$ of M having Σ as the collection of compact leaves such that the restriction of the foliation to $M \smallsetminus \Sigma$ is a fibration over S^1. In particular, $\mathcal{F}$ is a taut, cooriented, depth one foliation of M.

Our work here is partially motivated by understanding the interaction between Thurston's foliations and the canonical pseudo-Anosov suspension flow $\varphi_{\mathcal{C}}$. First, after replacing $\varphi_{\mathcal{C}}$ with a dynamic blowup φ (see Section 2.2), which modifies $\varphi_{\mathcal{C}}$ only at its singular orbits, we may isotope Σ so that it is positively transverse to φ. This is a consequence of our Strong Transverse Surface Theorem (Theorem 2.3), which strengthens a result of Mosher that established the existence of such a transverse surface in the homology class $[\Sigma]$.

Next, for any cross section S of φ (i.e. any fiber surface in the associated fibered cone, up to isotopy) we can spin S about Σ to obtain a infinite-type surface L that is transverse to φ, meets every flow line infinitely often in the forwards and backwards direction, and accumulates only on Σ; see Section 2.3. Hence, there is a well-defined first return map $f\colon L \to L$ along the flow φ; we call any such map obtained by this construction **spun pseudo-Anosov** (see Definition 3.1 for a more formal definition). It follows immediately that $M \smallsetminus \Sigma$ can be identified with the mapping torus of f and hence inherits a foliation by L–fibers. The foliation $\mathcal{F}$ of M is then obtained by adjoining the components of Σ as leaves. By construction, the foliation $\mathcal{F}$ is transverse to φ.

The spun pseudo-Anosov map $f\colon L \to L$ inherits additional structure from the flow φ. Since φ has invariant unstable/stable singular foliations $W^{u/s}$, their intersections $\mathcal{W}^{u/s} = L \cap W^{u/s}$ are a pair of f–invariant singular foliations of L. Moreover, the expanding/contracting dynamics of φ along its periodic orbits impose corresponding dynamics at the periodic points of f (see Section 4 for details).

This structure is the basis for the properties of spA maps listed above (and more formally detailed below).

Motivation from big mapping class groups. There has been recent interest in the study of big mapping class groups, i.e. mapping class groups of infinite-type surfaces. A significant component of this is understanding the extent to which there is a Nielsen–Thurston—like classification of elements of big mapping class groups (see e.g. [MCG, Problem 1.1]). From this point of view, spun pseudo-Anosov maps of infinite-type surfaces, at least for surfaces with finitely many ends, offer such a normal-form representative for atoroidal endperiodic maps. For such maps, the second item of Theorem C answers [MCG, Problem 1.5].

We also remark that atoroidal, endperiodic maps are central to recent work that relates the hyperbolic geometry of their mapping tori to combinatorial invariants [FKLL21, FKLL23].

1.2. **Main results.** Inspired by Nielsen–Thurston theory for finite-type mapping classes, one could ask which homeomorphisms $L \to L$ are isotopic to spA maps. Of course, such homeomorphisms must be atoroidal since their mapping torus embed in closed hyperbolic manifolds (i.e. the manifold M in Definition 3.1). Our complete answer is that this obvious necessary conditions are also sufficient:

Theorem A. *An endperiodic homeomorphism $g\colon L \to L$ is isotopic to an spA representative $f\colon L \to L$ if and only if it is atoroidal.*

See Theorem 3.2 and Theorem 3.3.

The proof shows how to construct, given the compactified mapping torus N of g, a closed hyperbolic manifold M with a pseudo-Anosov suspension flow φ and an embedding $N \to M$ that maps L to a leaf of a depth one foliation $\mathcal{F}_L$ of M that is transverse to a dynamic blowup of φ. The required spA map $f\colon L \to L$ is then the first return map to the leaf L of $\mathcal{F}_L$.

The construction of M and φ are fairly flexible and this leads to various strengthenings of Theorem A. In particular (from Theorem 4.2),

Theorem B. *An atoroidal endperiodic homeomorphism $g\colon L \to L$ is isotopic to an spA representative $f\colon L \to L$ which is honestly transverse to the pseudo-Anosov suspension flow.*

Here, 'honestly transverse' means that the surface is actually transverse to the flow, rather than only *almost* transverse. That is, one can vary the construction so that no dynamic blowup is needed. We call such a map "spA$^+$" – see Definition 3.4.

For any homeomorphism $g\colon L \to L$, we define its **growth rate** $\lambda(g)$ as the exponential growth rate of its periodic points:

$$\lambda(g) = \limsup_{n\to\infty} \sqrt[n]{\#\mathrm{Fix}(g^n)},$$

where $\mathrm{Fix}(g)$ denotes the set of fixed points of g. The following theorem gives various characterizations of the growth rate of an spA map, generalizing what is known about pseudo-Anosov homeomorphisms of finite-type surfaces. In its statement, $i(\alpha, \beta)$ denotes the geometric intersection number of curves α and β.

Theorem C (Characterizing stretch factors)**.** *Let $f\colon L \to L$ be spA. Then*

(1) $\lambda(f) = \inf_{g\simeq f} \lambda(g)$, *over all homotopic endperiodic maps g.*
(2) $\lambda(f) = \max_{\alpha,\beta} \limsup_{n\to\infty} \sqrt[n]{i(\alpha, f^n(\beta))}$, *where α, β are essential simple closed curves on L.*
(3) $\log \lambda(f)$ *is the topological entropy of the restriction of f to the (unique) largest invariant compact set.*

See Theorem 5.1, Corollary 6.2, and Theorem 6.4.

Because of the parallels drawn by Theorem C between $\lambda(f)$ and the stretch factor of a pseudo-Anosov surface homeomorphism, we call $\lambda(f)$ the **stretch factor** of the spA map f.

We note here that the first item in Theorem C follows from a stronger result: if $g\colon L \to L$ is endperiodic and isotopic to the spA map f, then for any f–periodic point of period n there is a Nielsen equivalent g–periodic point also of period n. See Section 6.1 for details. In this sense, spA maps have the tightest dynamics among isotopic endperiodic maps.

Foliation cones and entropy. Just as a closed manifold M might fiber in various ways, so too might a sutured manifold N—in an appropriate sense. Representing these foliations in $H^1(N)$ leads to the foliation cones of Cantwell–Conlon. In more detail, a class in $H^1(N)$ is **foliated** if it is dual to a fibration $N \smallsetminus \partial N \to S^1$ whose foliation by fibers extends to N by adjoining ∂N. Such a foliation is called a *depth one foliation suited to* N; see Section 2.1.

Cantwell and Conlon prove that the foliated classes in $H^1(N)$ comprise the integer points of a union of finitely many open rational polyhedral cones, each of which is called a **foliation cone** of N [CC99, CCF19]. This can also be recovered as a consequence of our next result:

Theorem D (All foliations are spun)**.** *There is a closed hyperbolic 3–manifold M and an embedding $i\colon N \to M$ such that the morphism $i^*\colon H^1(M) \to H^1(N)$ maps each fibered cone of M whose boundary contains $[\partial_+ N]$ onto a foliation cone of N, and each foliation cone of N is obtained in this manner.*

Consequently, each depth one foliation suited to N is obtained by spinning fibers of M around $\partial_\pm N$.

Moreover, for a fixed foliation cone $\mathcal{C}$ of N there is a semiflow $\varphi_{\mathcal{C}}$, obtained by restricting a pseudo-Anosov suspension flow on M, so that every depth one foliation $\mathcal{F}$ suited to N with $[\mathcal{F}] \in \mathcal{C}$ is transverse to $\varphi_{\mathcal{C}}$.

See Theorem 7.1 and Corollary 7.2.

Having organized the various ways that N can fiber into a finite collection of polyhedral cones and having defined stretch factors associated to each primitive integral point in these cones, we turn to explain how the stretch factors vary as the foliation is deformed (see Theorem 7.3):

Theorem E (Entropy)**.** *For each foliation cone $\mathcal{C}$, there is a continuous, convex function* $\mathrm{ent}\colon \mathcal{C} \to [0,\infty)$*, such that for any foliated class $[L] \in \mathcal{C}$,*

$$\mathrm{ent}([L]) = \log\left(\textit{stretch factor of the spun pseudo-Anosov } f_L\right).$$

As previously remarked, in the classical setting of a closed fibered manifold, the associated entropy function has the additional features that it is real analytic and blows up at the boundary of the cone. Interestingly, neither of these stronger properties need to hold here since essential annuli in N (which arise from invariant curves/lines of the monodromy $L \to L$) can create obstructions.

By combining Theorem E with previous work of the authors [LMT21], one also sees additional connections between the entropy functions on a fibered cone and the entropy function on the foliation cone obtained by cutting along a transverse surface. For details, see Section 7.3.

Invariant laminations and Handel–Miller theory. We conclude by explaining the connection to a previous approach to the study of endperiodic maps. In the early 1990s, Handel and Miller developed a theory to understand endperiodic maps using representatives that fix a canonical pair of invariant geodesic laminations $\Lambda^\pm_{\mathrm{HM}}$ with respect to a fixed hyperbolic metric, analogous to the Casson–Bleiler [CB88] approach to pseudo-Anosov theory in the finite-type setting. Although this was not written down by the authors, various expositions can be found in work of Fenley [Fen92, Fen97] and Cantwell–Conlon [CC99], with the definitive treatment appearing in [CCF19]. See Section 8 for more background.

The last theorem states that for any spA map $f\colon L \to L$, singular versions of the Handel–Miller laminations $\Lambda^\pm_{\mathrm{HM}}$ appear as sublaminations of its invariant foliations $\mathcal{W}^{u/s}$ and in this sense every spA map is a 'singular' Handel–Miller representative (see Theorem 8.4).

Theorem F (spAs are HM)**.** *The invariant singular foliations $\mathcal{W}^{u/s}$ of an spA map $f\colon L \to L$ contain invariant singular sublaminations $\Lambda^\pm$ which determine the same endpoints in the hyperbolic boundary of $\widetilde{L}$ as the Handel–Miller laminations $\Lambda^\pm_{\mathrm{HM}}$.*

The sublaminations $\Lambda^\pm$ of $\mathcal{W}^{u/s}$ are topological invariants of the isotopy class of f and are important for computing the topological entropy of f, as in Section 5. For a more precise statement of Theorem F and further details, see Section 8. In this sense, spA maps give an alternative, independent structure theory of atoroidal endperiodic maps that relies on the study of pseudo-Anosov flows rather than Handel–Miller theory.

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2. Background

Here we briefly collect some background needed for working with endperiodic maps and pseudo-Anosov flows.

2.1. **Endperiodic maps, depth one foliations, and sutured manifolds.** Let L be an orientable, connected, infinite-type surface with finitely many ends and no boundary. If $g\colon L \to L$ is a homeomorphism, an end e of L is **attracting** (or positive) if it has a neighborhood $U \subset L$ such that for some $n \geqslant 1$, $g^n(U) \subset U$ and $\bigcap_{i\geqslant 1} g^{in}(U) = \varnothing$. The end e is **repelling** (or negative) if it is attracting for g^{-1}. We say that g is **endperiodic** if each of its ends is either attracting or repelling. Throughout this paper, we also require surfaces to have no planar ends.

As in Section 1, if g has no invariant essential finite multicurve up to isotopy, then it is said to be **atoroidal**. The following lemma implies a simple property of atoroidal maps that will be useful later.

Lemma 2.1. *Let $g\colon L \to L$ be an endperiodic map, and let α be an essential closed curve in L. Suppose that each element of $\{\alpha, g(\alpha), g^2(\alpha), \dots\}$ is homotopic into a fixed compact subset of L. Then g is not atoroidal.*

Proof. Fix a hyperbolic metric on L and let α_i be the geodesic tightening of $g^i(\alpha)$. Say that a compact subsurface S of L with geodesic boundary is a "g-sink for α" if all α_i beyond some point lie in S. The hypothesis guarantees such a surface exists, and it is evident that if S is a g-sink then so is $g(S)$ after tightening its boundary to geodesics.

The intersection of two g-sinks, after tightening the boundary, is a g-sink. It follows that a minimal g-sink exists, which we call K. Since K is minimal, $K \subset g(K)$ up to isotopy. But since they are homeomorphic they are isotopic. The boundary of K then provides the invariant multicurve that shows g is not atoroidal. □

Compactified mapping tori of endperiodic maps. For our purposes in this article, a **sutured manifold** is a compact, oriented 3-manifold N whose boundary components each have a coorientation (this would be known elsewhere as a sutured manifold (M, γ) for which $\gamma = \varnothing$). We write $\partial N = \partial_+ N \sqcup \partial_- N$, where $\partial_+ N$ denotes the boundary components that are cooriented out of N and $\partial_- N$ denotes the boundary components that are cooriented in to N. The terminology of sutured manifolds will be convenient for us, but we will not use much from the theory.

Let N be a sutured manifold, let $\mathcal{F}$ be a cooriented codimension one foliation of N, and let $\mathcal{F}_0$ denote its set of compact leaves. The foliation $\mathcal{F}$ has **depth one** if $\mathcal{F}|_{N \smallsetminus \mathcal{F}_0}$ defines a fibration of $N \smallsetminus \mathcal{F}_0$ over S^1. When $\partial N \neq \varnothing$, all foliations of N in this article will be **suited to** N in the sense that $\partial N = \mathcal{F}_0$, as cooriented surfaces. In general, every depth one foliation we will consider is **taut** in the sense that each leaf is met by a compatibly oriented transverse curve or properly embedded arc.

The following theorem appears as [CC17, Proposition 6.21], see also [CC93, Theorem 1.1]. For the statement, note that a depth one foliation $\mathcal{F}$ suited to a sutured manifold N induces a *dual class* in $H^1(N)$, i.e. the class determined by the fibration $N \setminus \partial N \to S^1$.

Theorem 2.2 (Cantwell–Conlon)**.** *Let $\mathcal{F}$ and $\mathcal{F}'$ be taut, depth one foliations suited to the sutured manifold N inducing the same class on $H^1(N)$. Then $\mathcal{F}$ and $\mathcal{F}'$ are isotopic via a continuous isotopy that is smooth on $N \setminus \partial_\pm N$ and constant on $\partial_\pm N$.*

There is an important correspondence between depth one foliations suited to sutured manifolds and *compactified* mapping tori of endperiodic maps that we now describe. For more see [Fen97, Section 3], [CCF19, Lemma 12.5], or for an even more detailed treatment [FKLL21, Section 3].

Let $g\colon L \to L$ be endperiodic. The mapping torus of g is $L \times [0,1]/(x,1) \sim (f(x),0)$. This mapping torus comes equipped with an oriented 1-dimensional foliation, called the *suspension flow*, induced by the $\{p\} \times [0,1]$ foliation on $L \times [0,1]$; we refer to the leaves of this foliation as orbits of the suspension flow.

The mapping torus is noncompact, but it has a natural compactification obtained by appending an ideal point to each end of an orbit of the suspension flow that escapes compact sets. It follows from endperiodicity that g acts properly discontinuously on the sets of points in L that escape compact sets under iteration of g, and similarly for iterations of g^{-1}. From this one can see that the union of the ideal points is naturally a disconnected surface with one component for each g-orbit of an end of L. After gluing on this surface we obtain a compact manifold with boundary called the **compactified mapping torus** of g, which we denote by N_g.

The components of ∂N_g come with natural coorientations: components corresponding to negative ends of orbits are cooriented inward, and those corresponding to positive ends of orbits are cooriented outward. The unions of outward and inward boundary components are denoted $\partial_+ N_g$ and $\partial_- N_g$. This gives N_g the structure of a sutured manifold.

An important point for us will be that g is atoroidal if and only if N_g is atoroidal [FKLL21, Lemma 3.4]. Since no end of L is planar, each component of $\partial_\pm N_g$ has genus at least 2.

There is a natural cooriented depth one foliation $\mathcal{F}_L$ on N_g whose leaves are the noncompact (i.e. depth one) surfaces $L \times \{t\}$ as well as the components of $\partial_\pm N_g$. Informally, the positive ends of L spiral around $\partial_+ N_g$ and the negative ends spiral around $\partial_- N_g$. This foliation is taut [FKLL21, Lemma 3.3], so in particular $\partial_+ N_g$ and $\partial_- N_g$ are taut in the sense that they are Thurston norm-minimizing in $H_2(N_g)$ [Thu86].

2.2. **Pseudo-Anosov flows and dynamic blowups.** In this article, we consider only pseudo-Anosov flows φ on a 3-manifold M that are **circular**, i.e. equal to the suspension flow of a pseudo-Anosov homeomorphism up to reparametrization. One advantage here is that all of the structure we need concerning φ follows from the well-known structure of pseudo-Anosov homeomorphisms; see [Fri79]. For example, the unstable/stable invariant foliations $W^{u/s}$ of φ are simply the suspensions of the unstable/stable foliations of the pseudo-Anosov monodromy.

When dealing with a pseudo-Anosov flow φ it can be useful to slightly weaken the notion of what it means for an oriented surface Σ to be positively transverse to φ, obtaining the concept called "almost transversality." For this we briefly discuss dynamically blowing up singular orbits. Dynamic blowups and almost transversality were introduced by Mosher in [Mos90].

A flow $\varphi^\sharp$ is a **dynamic blowup** of φ if it is obtained by perturbing φ in a neighborhood of some of the singular orbits of φ, in the following way. We replace a singular orbit γ by the suspension A of a homeomorphism g of a finite tree T, i.e. $A = T \times I/g$. We can identify T with the intersection of A with a cross-sectional disk D. For an edge e with period n, g^n should fix the endpoints of e and act without fixed points on $\mathrm{int}(e)$. The edges of T, together with the intersection with D of the singular leaves containing γ, give a larger tree $\overline{T}$ in D. We orient each edge e in $\overline{T}$ according to the direction in which points are moved upon first return to e. We require that around each vertex of $\overline{T}$, these orientations alternate between outward and inward. See Figure 1.

The suspension A is a $\varphi^\sharp$-invariant union of annuli, called an annulus complex, and $\varphi^\sharp$ is semiconjugate to φ by a map that collapses A to γ and is otherwise one-to-one. The vector fields generating φ and $\varphi^\sharp$ differ only inside a small neighborhood of the orbits that were blown up. Each annulus in the complex A is called a **blown annulus**. The two boundary components of a blown annulus are periodic orbits of $\varphi^\sharp$. Inside the annulus, orbits of $\varphi^\sharp$ spiral from one boundary component to the other.

The **stable foliation** of $\varphi^\sharp$ is a singular foliation whose leaves are exactly the preimages of leaves of the stable foliation of φ under the semiconjugacy collapsing the various annulus complexes. Similarly for the **unstable foliation** of $\varphi^\sharp$. Thus if H is a stable leaf of φ containing a singular orbit γ of φ which is blown up to obtain $\varphi^\sharp$, then the stable leaf of $\varphi^\sharp$ corresponding to H will contain every blown annulus collapsing to γ. As such the stable and unstable leaves of $\varphi^\sharp$ affected by the dynamic blowup will be tangent along shared blown annuli (see Figure 1).

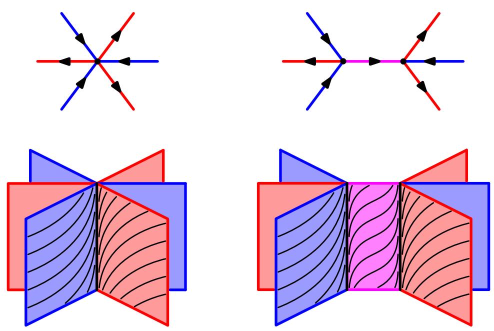

FIGURE 1. A simple dynamic blowup of a 3-pronged singular orbit. On the top we see the intersection with a transverse disk before and after the blowup. On the bottom we see the intersection with a 3-ball before and after the blowup. In this case the associated annulus complex is a single annulus, shown in pink, which belongs to both the stable and the unstable leaf shown. Dynamic blowups can be more combinatorially complicated than this; see for example Figure 2.

We say that Σ is **almost transverse** to φ if there exists a dynamic blowup $\varphi^\sharp$ of φ such that Σ is positively transverse to $\varphi^\sharp$. We also say that $\varphi^\sharp$ is an **almost pseudo-Anosov flow**. In other words, if Σ is transverse to $\varphi^\sharp$, then Σ is almost transverse to φ.

Suppose that Σ is a compact surface positively transverse to an almost pseudo-Anosov flow $\varphi^\sharp$, obtained by dynamically blowing up a pseudo-Anosov flow φ. Let a be a blown annulus of $\varphi^\sharp$. Since $\Sigma \cap a$ is compact it consists of either a collection of arcs between components of ∂a or a collection of circles in $\mathrm{int}(a)$ (see Figure 2). If $\Sigma \cap a$ consists of arcs or is empty, then a can be collapsed to obtain a "less blown up" flow to which Σ is also transverse. Hence we say that $\varphi^\sharp$ is **minimally blown up** with respect to Σ if for every blown annulus a, the intersection of Σ with a is a non-empty union of circles.

Similarly, if $\mathcal{F}$ is a depth one foliation transverse to $\varphi^\sharp$, we say $\varphi^\sharp$ is **minimally blown up** with respect to $\mathcal{F}$ if there is no blown annulus a of $\varphi^\sharp$ such that $\mathcal{F}|_a$ is a foliation by properly embedded arcs. We note that $\varphi^\sharp$ is minimally blown up for $\mathcal{F}$ if and only if it is minimally blown up for $\mathcal{F}_0$. Indeed, if a is a blown annulus and $\mathcal{F} \cap a$ is not a foliation by arcs, then it contains a closed leaf, which must be a component of $\mathcal{F}_0 \cap a$, since $\mathcal{F}$ is depth one.

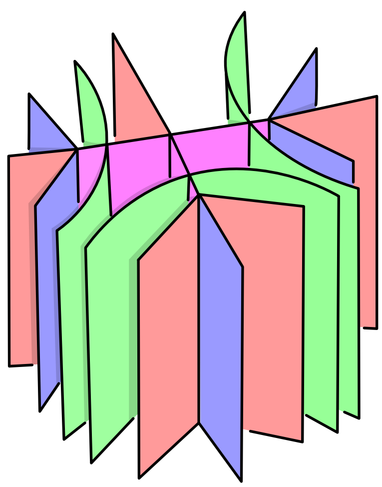

FIGURE 2. A local picture of a dynamically blown up 5-pronged singular orbit, together with part of a surface (green) transverse to the blown up flow. In this case the intersection of the surface with the blown annuli will be 3 circles.

A key step in our construction of spA maps uses the following theorem, which is a special case of the main result in [LMT24, Theorem A].

Theorem 2.3 (Strong transverse surface theorem)**.** *Let M be a closed oriented 3–manifold with a circular pseudo-Anosov flow φ. For an oriented surface $\Sigma \subset M$, Σ is almost transverse to φ, up to isotopy, if and only if Σ is taut and has nonnegative intersection number with every closed orbit of φ.*

The "if" statement of Theorem 2.3 is the more difficult one and is proved as follows. There is a *veering triangulation* associated to φ, whose 2-skeleton is a cooriented branched surface in $M - U$, where U is a small regular neighborhood of the singular orbits of φ. This is a *partial branched surface in* M, in the sense of [Lan22] (see Section 6.2.1). If Σ is a taut surface pairing nonnegatively with the closed orbits of φ, then one can apply the techniques of that paper to isotope Σ to be carried by the partial branched surface; this means it is carried in $M - U$ (in the normal sense of branched surfaces), and intersects U in a controlled way. The partial branched surface is positively transverse to φ ([LMT21, Theorem 5.1]) and its intersection with ∂U is easily understandable. This then allows us, given the data of $\Sigma \cap U$, to find a dynamic blowup of φ transverse to Σ.

2.3. **The spinning construction.** We describe a special case of a standard construction in foliations which appears as [Cal07, Example 4.8]. We will refer to this operation as **spinning** (note that Calegari refers to it as "spiraling").

Let M be a closed, oriented 3-manifold equipped with a flow φ. Suppose that $S \subset M$ is a cross section to φ, meaning that it is a closed oriented surface positively transverse to φ and intersecting each orbit. A consequence of this is that $M \mathbin{\backslash\backslash} S$ is homeomorphic to $S \times I$, where we can take the I-fibers to be the foliation on $M \mathbin{\backslash\backslash} S$ induced by the orbits of φ. Further suppose that $\Sigma \subset M$ is another closed oriented surface which is transverse to both S and φ such that no component of Σ is isotopic to a component of S.

Let $\Sigma \times [-1, 1]$ be a small tubular neighborhood of Σ in M, where each $\{p\} \times [-1, 1]$ fiber is an orbit segment of φ. Let

$$\overline{\Sigma} = \bigcup_{|n|>0} \Sigma \times \left\{\frac{1}{n}\right\}.$$

Let L be the oriented cut and paste sum of $S \smallsetminus \Sigma$ with $\overline{\Sigma}$, smoothed so as to be transverse to φ.

Then L is a noncompact surface. Moreover, $(M \smallsetminus \Sigma) \mathbin{\backslash\backslash} L$ is homeomorphic to $L \times [0, 1]$ where each $\{p\} \times [0, 1]$ is an orbit segment of φ. Hence we can fill in $M - (\Sigma \cup L)$ with a product foliation $L \times (0, 1)$ to obtain a foliation $\mathcal{F}$ of M. The fact that S is a cross section with no component isotopic to a component of Σ implies that $S \cap \Sigma$ is homologically nontrivial in each component of Σ. This in turn implies that L has ends that "spiral" around the components of Σ, the components of Σ are the only compact leaves of $\mathcal{F}$, and the noncompact leaves of $\mathcal{F}$ define a fibration of $M \smallsetminus \Sigma$ over S^1. Hence $\mathcal{F}$ is a taut depth one foliation of M.

Note that $N = M \mathbin{\backslash\backslash} \Sigma$ is naturally a sutured manifold, with the components of $\partial_\pm N$ cooriented by φ, and $\mathcal{F}$ induces a taut, depth one foliation suited to N. We often continue to refer to this foliation of N by $\mathcal{F}$ and also say that it is the result of spinning $N \cap S$ about $\partial_\pm N$.

3. Existence of spA representatives

We begin by formally defining the main object of the paper.

Definition 3.1 (spun pseudo-Anosov)**.** *An endperiodic map $f : L \to L$ is* ***spun pseudo-Anosov*** *(or* ***spA****) if there exists a depth one foliation $\mathcal{F}$ of a hyperbolic 3-manifold M and a transverse, circular almost pseudo-Anosov flow φ that is minimally blown up with respect to $\mathcal{F}$ such that L is a leaf of $\mathcal{F}$ and f is a power of the first return map induced by φ.*

Definition 3.1 includes the maps described in Section 1.1 that were obtained by spinning a pseudo-Anosov homeomorphism of a finite-type surface, and the spA maps we produce in this section will come from a generalization of this construction. See, e.g., Remark 3.14.

The main theorem of this section gives the "if" direction of Theorem A:

Theorem 3.2 (spA representatives exist)**.** *Each atoroidal, endperiodic map is isotopic to a spun pseudo-Anosov map.*

Theorem 3.2 will follow immediately from a more detailed result (Theorem 3.3), which we now turn to state. Let $g : L \to L$ be an endperiodic map and let N_g be its compactified mapping torus. Recall that by construction N_g comes equipped with a depth one foliation

$\mathcal{F}_L$ whose compact leaves are $\partial_\pm N_g$ and whose depth one leaves are parallel to L. We call $\mathcal{F}_L$ the **depth one foliation associated to** L.

Theorem 3.3. *Let $g\colon L \to L$ be an atoroidal, endperiodic map with compactified mapping torus $N = N_g$. Then there exists*

- *a hyperbolic fibered 3-manifold M,*
- *a pseudo-Anosov suspension flow $\varphi^\flat$ on M, and*
- *an embedding $N \to M$ and a dynamic blowup φ of $\varphi^\flat$ so that the depth one foliation $\mathcal{F}_L$ extends to a depth one foliation of M with respect to which φ is transverse and minimally blown up.*

Hence, the first return map $f\colon L \to L$ determined by φ is a spun pseudo-Anosov map isotopic to g.

The spA map $f\colon L \to L$ given by Theorem 3.3 is called an **spA representative** of g. We note that the constructed manifold M and flow $\varphi^\flat$ is quite flexible and this is one strength of the theory. For example, let us also give a natural strengthening of Definition 3.1:

Definition 3.4 (spA$^+$)**.** *An endperiodic map $f\colon L \to L$ is **spA**$^+$ if there exists a circular pseudo-Anosov flow φ on a hyperbolic 3-manifold M and a depth one foliation $\mathcal{F}$ transverse to φ such that L is a depth one leaf of $\mathcal{F}$ and f is a power of the first return map induced by φ.*

That is, a map $f\colon L \to L$ is spA$^+$ if no dynamic blowup is needed to make $\mathcal{F}$ positively transverse to φ. We will prove in Theorem 4.2 that every atoroidal endperiodic map is isotopic to an spA$^+$ map by showing that M in Theorem 3.3 (or in Definition 3.1) can be taken so that $\mathcal{F}$ is transverse to a circular pseudo-Anosov flow.

Remark 3.5 (The spA package)**.** Each spA map $f\colon L \to L$ comes equipped with the manifold M, foliation $\mathcal{F}$, and flow φ as in Definition 3.1. Moreover, the compactified mapping torus $N = N_f$ can also be recovered (as in Theorem 3.3) as the component of $M \mathbin{\backslash\!\backslash} \mathcal{F}_0$ containing the leaf L, where $\mathcal{F}_0$ denotes the compact (depth zero) leaves of $\mathcal{F}$. In general, the resulting map $N \to M$ could identify components of $\partial_\pm N$, but one can modify the foliation $\mathcal{F}$ by replacing its compact leaves with standardly foliated I-bundles. After this modification, $N \to M$ is an embedding. We always assume that this is the case and call the data: $M, \mathcal{F}, \varphi, N$ an **spA package** for $f\colon L \to L$.

3.1. **Juncture classes and spiraling neighborhoods.** Fix an endperiodic map $g\colon L \to L$. Let $N = N_g$ be its compactified mapping torus with the associated taut depth one foliation $\mathcal{F} = \mathcal{F}_L$. Let $\mathcal{L} = \mathcal{L}_g$ be the transverse 1-dimensional oriented foliation on N induced by the suspension flow of g. We will sometimes refer to $\mathcal{L}$ as a semiflow on N. Note that $\mathcal{F}$ is cooriented by $\mathcal{L}$.

The foliation $\mathcal{F}$ uniquely determines its dual class $\xi \in H^1(N)$ that assigns to each oriented loop its signed intersection number with L. The pullback of ξ to $H^1(\partial N)$ is called the **juncture class** $j \in H^1(\partial N)$ of ∂N associated to $\mathcal{F}$, and the restriction of j to each component Σ of ∂N is called the juncture class of Σ. A **realization** of the juncture class of Σ is a cooriented collection of essential simple curves in Σ that represent the juncture class in H^1 having the property that no subset is the boundary of a complementary subsurface of Σ with either the outward or inward pointing coorientation.

Remark 3.6. Since Σ is closed, a juncture class (or any cohomology class) can be realized by a single cooriented curve.

A **spiraling neighborhood** U of a component Σ of $\partial_\pm N$ is a collar neighborhood of Σ, foliated by arcs of $\mathcal{L}$, whose outer boundary component is Σ and whose inner boundary component is a surface Σ_U that is transverse to both $\mathcal{L}$ and L. See, for example, [CC93, Section 3] or [Fen92, Section 4]. Note that $\Sigma_U \cap L$ represents the juncture class on Σ after identifying Σ_U with Σ along the $\mathcal{L}$ fibers and coorienting using the coorientation on L. Moreover, after an isotopy of Σ_U to remove product regions between Σ_U and L, we may assume that $\Sigma_U \cap L$ realizes the juncture class on Σ. In general, a spiraling neighborhood is a disjoint collection of spiraling neighborhoods of each component of $\partial_\pm N$.

Given a spiraling neighborhood U of $\partial_\pm N$ we can collapse it along fibers of $\mathcal{L}$ so that $L \setminus \mathrm{int}(U)$ is sent to a properly embedded, cooriented surface $L_U \subset N$ whose boundary, with its induced coorientation, represents the juncture class in each component of $\partial_\pm N$. More precisely, L_U is obtained from $L \setminus \mathrm{int}(U)$ by flowing $L \cap \Sigma_U$ into $\partial_\pm N$ within U and slightly isotoping the resulting surface rel boundary towards $\partial_\pm N$ so that its interior is transverse to $\mathcal{L}$. See Figure 3.

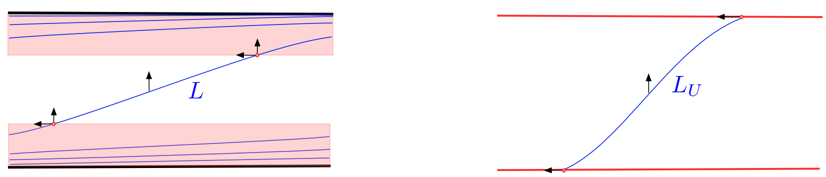

FIGURE 3. Producing L_U by pushing the spiraling of L into $\partial_\pm N$.

By construction, L_U is dual to ξ and its interior is positively transverse to $\mathcal{L}$. One should think of L_U as obtained from L by 'pushing the spiraling into $\partial_\pm N$.' The spiraling neighborhood U determines how much of the spiraling is pushed into $\partial_\pm N$ and the following well-known lemma says that by choosing U appropriately, any realization of the juncture class j occurs as the boundary of L_U. See for example [Gab87, Lemma 0.6] where a more general statement is proven.

Lemma 3.7. *For any realization m of the juncture class $j \in H^1(\partial_\pm N)$, there is a spiraling neighborhood U so that $\partial L_U = m \subset \partial_\pm N$.*

Finally, we say that the properly embedded surface L_U is a **prefiber** if its interior meets each orbit of the semiflow $\mathcal{L}$. It is easy to see that for any L_U there is a spiraling neighborhood $U' \subset U$ so that $\partial L_{U'} = \partial L_U$ and $L_{U'}$ is a prefiber. Informally, $L_{U'}$ can be obtained by "peeling off" another layer of L from each component of $\partial_\pm N$.

3.2. **Extensions to the h–double.** Let $h \colon \partial_\pm N \to \partial_\pm N$ be a component-wise homeomorphism. The h–**double** of N is the manifold $M = M(N, h)$ obtained by taking two copies of N and gluing their boundaries together by h. When considering the triple $(N, \mathcal{F}, \mathcal{L})$ as above, M comes equipped with an induced taut depth one foliation and transverse, oriented 1-dimensional foliation, still denoted by $\mathcal{F}$ and $\mathcal{L}$ respectively, that extend the associated foliations on the two copies of N.

Let us be more precise about the construction of the h–double. Let $N^\downarrow$ denote the sutured manifold N with the coorientation on $\partial_\pm N$ reversed. Hence, there are identifications $\sigma_\pm \colon \partial N_\pm \to \partial N^\downarrow_\mp$ induced by the identity $N \to N^\downarrow$. The sutured manifold $N^\downarrow$ comes equipped with a depth one foliation $\mathcal{F}^\downarrow$ and oriented one dimensional foliation $\mathcal{L}^\downarrow$ whose coorientation and orientation, respectively, have been reversed. We then glue N to $N^\downarrow$

via the map $\sigma_\pm \circ h\colon \partial_\pm N \to \partial N^{\downarrow}_{\mp}$ and call the resulting manifold $M = M(N,h)$. By construction, the foliations $\mathcal{F}$ and $\mathcal{F}^{\downarrow}$ combine to give a taut, cooriented depth one foliation on M, which we continue to denote by $\mathcal{F}$, that is positively transverse to the oriented one dimensional foliation $\mathcal{L} \cup \mathcal{L}^{\downarrow}$, which we continue to denote by $\mathcal{L}$. We can smooth $\mathcal{L}$ in a neighborhood of $\partial_\pm N$ maintaining transversality to $\mathcal{F}$ and parameterize to obtain a flow on M that is positively transverse to $\mathcal{F}$.

Remark 3.8. The h–double $M = M(N,h)$ depends only on the sutured manifold N and h. Hence, any depth one foliation suited to N (and transverse semiflow) automatically extend to M by the construction.

We identify N with its image in M coming from the construction.

Proposition 3.9. *If $h\colon \partial_\pm N \to \partial_\pm N$ reverses the sign of each juncture class (i.e. $h^*j = -j$ in $H^1(\partial_\pm N)$), then for any spiraling neighborhood U of $\partial_\pm N$ for which L_U is a prefiber, there is a closed, cooriented surface S in $M = M(N,h)$ so that*

(1) S has positive transverse intersection with every orbit of the extended flow $\mathcal{L}$ on M,
(2) S transversely intersects $\partial_\pm N \subset M$, and
(3) $S \cap N$ is properly isotopic to L_U.

Remark 3.10. For Proposition 3.9, one can take h to be any orientation preserving homeomorphism whose restriction to each component of $\partial_\pm N$ acts by $-I$ on H_1. In particular a hyperelliptic involution has this property.

Proof. Fix a spiraling neighborhood U of $\partial_\pm N$ and consider the realization $m = \partial L_U$ of the juncture class j in $\partial_\pm N$. Observe first that

$$-j = \sigma^*_\pm(j^{\downarrow})$$

which follows from the fact that $\mathcal{F}^{\downarrow}$ is $\mathcal{F}$ with its coorientation reversed, i.e. under the identification $N \cong N^{\downarrow}$ the class in H^1 determined by $\mathcal{F}^{\downarrow}$ is exactly $-j$. Now the hypothesis on h implies that

$$h^* \circ \sigma^*_\pm(j^{\downarrow}) = h^*(-j) = j.$$

This implies that the cooriented image $m^{\downarrow} = \sigma_\pm \circ h(m)$ is a realization of the juncture class $j^{\downarrow}$.

Now by Lemma 3.7 there is a spiral neighborhood V of $\partial_\pm N^{\downarrow}$ so that the boundary of the properly embedded surface $L^{\downarrow}_V$ in $\partial N^{\downarrow}_\pm$ is equal to $m^{\downarrow}$ and so that L_V is a prefiber. Hence in the h–double M, the boundaries of L_U and $L^{\downarrow}_V$ are identified with compatible coorientations. We define S to be the unions $L_U \bigcup L^{\downarrow}_V$. Since the interiors of L_U and $L^{\downarrow}_V$ are each positively transverse to the extended flow $\mathcal{L}$ and their coorientations agree across their boundary, we see that S can be smoothed in a neighborhood of $\partial_\pm N$ to be positively transverse to $\mathcal{L}$ and $\partial_\pm N$. See Figure 4. Since L_U is a prefiber, S intersects each orbit of the extended flow $\mathcal{L}$ on M. $\square$

Since S has positive transverse intersection with each orbit of $\mathcal{L}$, the manifold $M \mathbin{\backslash\backslash} S$ is a product I-bundle foliated by segments of $\mathcal{L}$. This is to say that there is a well-defined first return map $S \to S$ that determines a fibration of M over S^1 with fiber S.

Remark 3.11 (Spinning S back to $\mathcal{F}$)**.** By construction, spinning the surface S obtained in Proposition 3.9 about $\partial_\pm N$ reproduces the foliation $\mathcal{F}$.

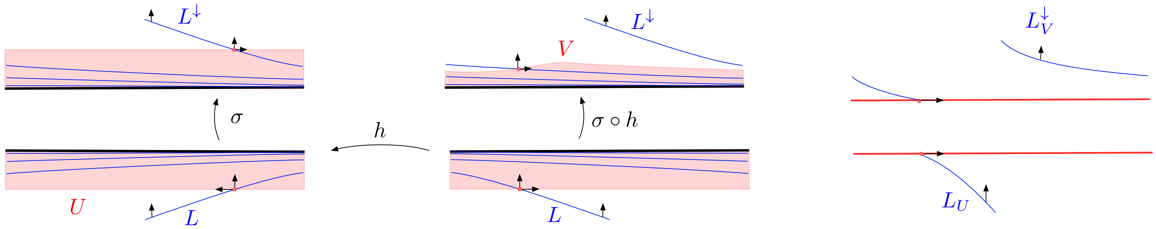

FIGURE 4. Producing the transverse surface $S = L_U \bigcup L_V^{\downarrow}$.

3.3. **Hyperbolic manifolds and circular pseudo-Anosov flows.** Recall that since $g\colon L \to L$ is atoroidal its compactified mapping torus $N = N_g$ is atoroidal and each component of $\partial_\pm N$ is a closed surface of genus at least 2.

Lemma 3.12. *For N as above, the gluing map h (satisfying Proposition 3.9) can be chosen so that $M(N, h)$ admits a hyperbolic structure.*

Proof. We first consider the special case where $N = S \times [0, 1]$. Here, if h_1 is the restriction of h to $\partial_+ N = S \times 1$ and h_0 is the restriction of h to $\partial_- N = S \times 0$, then $M(N, h)$ is the mapping torus of $h_0^{-1} h_1 \colon S \to S$, which is hyperbolic if and only if $h_0^{-1} h_1$ is pseudo-Anosov [Thu98, Ota01].

Otherwise, JSJ theory [Jac80] gives proper essential subsurfaces Y_+ of $\partial_+ N$ and Y_- of $\partial_- N$ such that each essential annulus A in N can be isotoped so that $\partial A \cap \partial_+ N \subset Y_+$ and $\partial A \cap \partial_- N \subset Y_-$. Hence, if h is chosen so that the image of any curve in $Y_\pm$ is not homotopic into $Y_\pm$, then $M(N, h)$ is atoroidal and hence hyperbolic by Thurston's hyperbolization theorem (see e.g. [Kap01]).

From this, it is easy to produce a gluing map h so that $M(N, h)$ is hyperbolic and for which Proposition 3.9 holds. For example, h can be taken so that on each component it is a hyperelliptic involution composed with a high power of a pseudo-Anosov homeomorphism that acts trivially on H_1 (i.e. is in the Torelli group). □

We now return to the context of Theorem 3.3, where $g\colon L \to L$ is endperiodic and atoroidal and $N = N_g$ is its compactified mapping torus. Let $h\colon \partial_\pm N \to \partial_\pm N$ be any component-wise homeomorphism that satisfies the conditions of Proposition 3.9 and Lemma 3.12 and let $M = M(N, h)$ be the associated h–double. As in Section 3.2, there is a fixed embedding $N \to M$ such that $\mathcal{F}(= \mathcal{F}_L)$ and $\mathcal{L}$ extend to M.

With this structure fixed, we turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. Let S be the properly embedded surface obtained from Proposition 3.9. Since S has positive transverse intersection with each orbit of $\mathcal{L}$ on M, S is a fiber in a fibration of M over the circle and the class $[S]$ is contained in the interior of the cone $\mathcal{C}_{\mathcal{L}} \subset H^1(M)$ of classes that are nonnegative on homology directions of $\mathcal{L}$ (see [Fri79] for details). Since $\partial_\pm N$ is also positively transverse to $\mathcal{L}$ by construction, $[\partial_\pm N]$ is also contained in $\mathcal{C}_{\mathcal{L}}$. Indeed, $[\partial_\pm N]$ is contained in the boundary of this cone unless N is itself a product. Note that since $\partial_\pm N$ is the union of compact leaves of the taut foliation $\mathcal{F}$, $\partial_\pm N$ is taut.

Since M is hyperbolic, the monodromy $S \to S$ is isotopic to a pseudo-Anosov homeomorphism. By suspending the pseudo-Anosov representative of this monodromy we obtain a pseudo-Anosov flow φ on M, unique up to isotopy and reparameterization, which is transverse to S.

According to Fried [Fri79, Theorem 14.11], the cone $\mathcal{C}_\varphi \subset H^1(M)$ of classes that are nonnegative on the homology directions for φ is equal to the closure of the fibered cone containing $[S]$, and $\mathcal{C}_{\mathcal{L}} \subset \mathcal{C}_\varphi$. Hence, we also have that $[\partial_\pm N]$ is in the cone $\mathcal{C}_\varphi$ and so $\partial_\pm N$ has nonnegative intersection number with each closed orbit of φ. Since $\partial_\pm N$ is also taut, Theorem 2.3 implies that $\partial_\pm N$ is almost transverse to φ, up to isotopy.

Let $\varphi^\sharp$ be the associated minimal dynamic blowup suitably isotoped so that it is positively transverse to $\partial_\pm N$. This isotopy carries S to a cross section S' of $\varphi^\sharp$. As in Remark 3.11, the foliation $\mathcal{F}$ of M is obtained by spinning S about $\partial_\pm N$. If we denote by $\mathcal{F}'$ the foliation of M obtained by spinning the $\varphi^\sharp$–cross section S' about the $\varphi^\sharp$–transverse surface $\partial_\pm N$, we obtain a transverse, taut, depth one foliation $\mathcal{F}'$ of M. The following lemma states that $\mathcal{F}$ and $\mathcal{F}'$ are isotopic. Once established, we can isotope M so that $\mathcal{F}$ is positively transverse to the flow $\varphi^\sharp$, thereby completing the proof of the theorem. □

It only remains to prove the following lemma, which follows easily from Theorem 2.2.

Lemma 3.13. *Let S and S' be isotopic fibers of M and suppose that Σ is a taut surface in the boundary of the associated fibered cone. Then the foliations $\mathcal{F}$ and $\mathcal{F}'$ obtained by spinning S and S' around Σ are isotopic in M.*

Proof. Let $N_1, \dots, N_k$ be the components of $M \backslash\backslash \Sigma$, each of which is a sutured manifold that comes equipped with two taut, depth one foliations $\mathcal{F}_i, \mathcal{F}_i'$. Since S' and S are isotopic, the classes in $H^1(N_i)$ associated to $\mathcal{F}_i$ and $\mathcal{F}_i'$ are equal for each i. Hence by Theorem 2.2, $\mathcal{F}_i$ is isotopic in N_i to $\mathcal{F}_i'$ via an isotopy that is constant on $\partial_\pm N_i$. Therefore, these isotopies glue together in M to give an ambient isotopy taking $\mathcal{F}$ to $\mathcal{F}'$, completing the proof. □

Remark 3.14 (Spinning and de-spinning)**.** The proof of Theorem 3.3 shows that the foliation $\mathcal{F}$ on the h–double M is obtained by spinning the cross section S along $\partial_\pm N$, up to isotopy. Similarly, the restricted foliations $\mathcal{F}$ on N is obtained by spinning the properly embedded surface $S \cap N$ about $\partial_\pm N$.

Reversing this process, the argument in Proposition 3.9 shows by choosing a spiraling neighborhood of each compact leaf of $\mathcal{F}$ in M (or, in other words, choosing spiraling neighborhoods of $\partial_\pm N$ in N and $N^\downarrow$) one can produce a (nonunique) surface S that is a cross section of φ. We call this process **de-spinning**.

We note that it is not the case that every depth one foliation of an arbitrary closed manifold can be de-spun to a fibration because of a basic cohomological obstruction: each compact leaf has two associated juncture classes from leaves spiraling on it from either side and the foliation can be de-spun if and only if these classes agree.

4. Multi sink-source dynamics

The goal of this section is to relate the periodic point behavior of an spA map $f : L \to L$ to the dynamics of the action of a lift $\widetilde{f}$ on the universal cover $\widetilde{L}$ and its compactification $\mathbb{D} = L \cup \partial\widetilde{L}$, defined with a suitable hyperbolic metric. The spA map comes with a pair of invariant foliations inherited from those of the pseudo-Anosov suspension flow, whose expanding and contracting half-leaves interact with the action on the circle at infinity (see below for complete definitions).

A homeomorphism $h\colon S^1 \to S^1$ is said to have **multi sink-source dynamics** if it has a finite number $k \geqslant 4$ of fixed points that alternate between attracting and repelling.

Theorem 4.1. *Let $f\colon L \to L$ be spA and let $\widetilde{f} : \widetilde{L} \to \widetilde{L}$ be a lift of f. Then $\widetilde{f}$ has a fixed point $\widetilde{p} \in \widetilde{L}$ if and only if it has a power $\widetilde{f}^n$ which acts on $\partial\widetilde{L}$ with multi sink-source dynamics. Moreover when this happens, $\widetilde{f}^n$ fixes the half-leaves at $\widetilde{p}$ and its attracting/repelling points are equal to the endpoints of expanding/contracting half-leaves.*

This generalizes well-known properties of pseudo-Anosov maps on compact surfaces, however there are a number of complications to deal with in our setting. Because the surface is only almost-transverse to the pseudo-Anosov flow, the structure of the stable and unstable foliations is harder to work with; in particular they are not everywhere transverse. The local dynamics on the circle at infinity require more work to understand, especially showing that sinks/sources on the circle are actually sinks/sources on the closed disk (see Lemma 4.9).

One application of this theorem is a proof of Theorem B, which we restate here:

Theorem 4.2. *Each atoroidal, endperiodic map is isotopic to an spA$^+$ map.*

Summary of the section: In Section 4.1 we analyze the behavior of *periodic half-leaves* of the stable and unstable foliations of L, by explaining the possibilities for their suspensions in the two-dimensional foliations of N. In Section 4.2 we recall results of Fenley and Cantwell-Conlon describing hyperbolic metrics on L and how they give the universal cover $\widetilde{L}$ a canonical circle at infinity, and on the quasi-geodesic properties of the foliation leaves in this metric.

In Section 4.3 we prove Lemma 4.9 and Corollary 4.10 which give the sink/source properties at infinity for endpoints of periodic half-leaves.

In Section 4.4 we prove Proposition 4.11, which gives one direction of Theorem 4.1.

In Section 4.5 we will apply what we have so far to prove Theorem 4.2 on spA$^+$ representatives. Finally, we will apply this in Section 4.6 to prove the other direction of Theorem 4.1, which will be stated in Proposition 4.15.

Throughout this section we fix a spun pseudo-Anosov map $f\colon L \to L$, together with an spA package (as in Remark 3.5) denoted by $M, \mathcal{F}, \varphi, N$, where $M, \mathcal{F}, \varphi$ are as in Definition 3.1 and $N \subset M$ is the compactified mapping torus of f. Moreover, we denote the induced semiflow on N as φ_N. For simplicity, we often denote the closed surface $\partial_\pm N$ in this section by $\partial_\pm$, and note that it comprises a subset of the compact leaves $\mathcal{F}_0$ of $\mathcal{F}$.

Since φ is minimally blown up with respect to $\mathcal{F}$, for each blown annulus A, $\mathcal{F}_0 \cap A$ is nonempty and consists of finitely many curves homotopic in M to the core of A. In particular, no closed orbit in the boundary of a blown annulus intersects $\mathcal{F}_0$. See Figure 5.

4.1. **Periodic half-leaves and annuli.** The circular flow φ has invariant stable and unstable singular foliations which we denote W^s and W^u, respectively. There is an induced semiflow φ_N on N; let W^s_N and W^u_N denote the foliations preserved by φ_N that arise by cutting W^s and W^u along $\partial_\pm$. Finally we define $\mathcal{W}^s = L \cap W^s$ and $\mathcal{W}^u = L \cap W^u$ to be the invariant **stable and unstable foliations** of $f\colon L \to L$. By construction, $\mathcal{W}^u$ and $\mathcal{W}^s$ suspend in f's compactified mapping torus N to be W^u_N and W^s_N, respectively.

If p is a periodic point of f then the half-leaves of $\mathcal{W}^u$ and $\mathcal{W}^s$ emanating from p are fixed by a power of f, and we call them **periodic half-leaves** of f. A periodic half-leaf contained in a blown up annulus is called a periodic **blown half-leaf**. We remark that $\mathcal{W}^u$ and $\mathcal{W}^s$ are transverse on L *except* at blown half-leaves, which are common to both foliations. Regardless, each periodic half-leaf ℓ of f is either **contracting** or **expanding** depending on whether iterating positive or negative powers of f attract points of ℓ to its periodic point p.

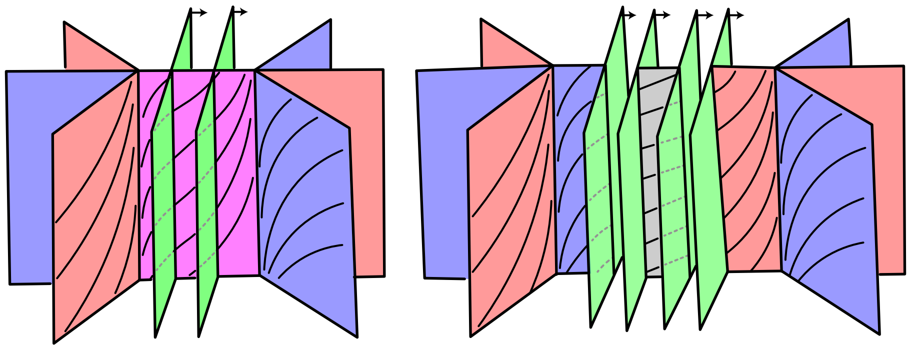

FIGURE 5. Left: two pieces of $\mathcal{F}_0$ (green) intersecting a blown annulus (pink). Right: the result of cutting along $\mathcal{F}_0$, with $\partial_+(M\backslash\backslash\mathcal{F}_0)$ and $\partial_-(M\backslash\backslash\mathcal{F}_0)$ indicated by the boundary coorientations. The blown annulus gives rise to two periodic half-leaves in $M\backslash\backslash\mathcal{F}_0$, both of which are technically stable *and* unstable. However, one of the half-leaves is expanding (blue, touching $\partial_+(M\backslash\backslash\mathcal{F}_0)$) and one is contracting (red, touching $\partial_-(M\backslash\backslash\mathcal{F}_0)$). Cutting along $\mathcal{F}_0$ also produces an annular leaf which touches $\partial_+(M\backslash\backslash\mathcal{F}_0)$ and $\partial_-(M\backslash\backslash\mathcal{F}_0)$ and is not periodic (gray).

The main goal of this subsection is Lemma 4.3, which constrains the asymptotic behavior of such half-leaves.

If e is an end of L we say that a subset $A \subset L$ **accumulates on** e if every neighborhood of e has non-empty intersection with A. We say that A **escapes** e if for every neighborhood U of e there is a compact $K \subset A$ such that $A \smallsetminus K \subset U$.

Lemma 4.3. *Every periodic half-leaf of f accumulates on some end of L. Expanding half-leaves accumulate on the positive ends, and contracting half-leaves accumulate on the negative ends. Moreover, each periodic blown half-leaf escapes a unique end.*

This lemma will follow easily once we develop the picture of the half-leaves of the 2-dimensional foliations $W_N^{s/u}$ obtained as suspensions of the periodic half-leaves of $\mathcal{W}^{s/u}$.

We say that a leaf of $W^{s/u}$ or $W_N^{s/u}$ is **periodic** if it contains a periodic orbit of φ or φ_N respectively. A **periodic half-leaf** of $W^{s/u}$ or $W_N^{s/u}$ is a component of $H_0\backslash\backslash\{\gamma_i\}$, where H_0 is a periodic leaf of $W^{s/u}$ or $W_N^{s/u}$ respectively, and $\{\gamma_i\}$ is the collection of all periodic orbits contained in H_0. A periodic half-leaf of $W^{s/u}$ is compact if and only if it is a blown annulus.

Let H_0 be a periodic leaf of $W^{s/u}$. If H_0 contains no blown annuli then it contains a unique periodic orbit γ. In this case $H_0\backslash\backslash\gamma$ is a union of noncompact periodic half-leaves, each a half-closed annulus adjacent to γ so that the flow lines in it spiral toward or away from γ if H_0 is in W^s or W^u, respectively. If H_0 contains blown annuli, then they are attached along their boundaries in a tree pattern, and attached to this complex are noncompact half-leaves (see Figure 2).

By definition, the periodic leaves of $W_N^{s/u}$ obtained from H_0 are the components of $H_0\backslash\backslash\partial_\pm N$ which contain a periodic orbit. Because φ is minimally blown up with respect to $\mathcal{F}$ by assumption, each blown annulus that meets N is cut by $\partial_\pm$ and hence each periodic leaf of $W_N^{s/u}$ contains a unique periodic orbit.

If H is a periodic half-leaf of $W_N^{u/s}$ then we say that H is **contracting** or **expanding** if every flow line in H is asymptotic to H's unique periodic orbit in the forward or backward direction, respectively. Periodic half-leaves that are contained in blown annuli (and hence are half-leaves of *both* W_N^u and W_N^s) can be expanding or contracting (see Figure 5). However, if any other periodic half-leaf is contracting or expanding, then it is contained in a leaf of W^s or W^u, respectively.

The next lemma describes the pieces obtained by cutting leaves of $W_N^{s/u}$ along periodic orbits, and shows in particular that each periodic half-leaf of $W_N^{s/u}$ is either expanding or contracting.

Lemma 4.4. *Let H_0 be a periodic leaf of W^u or W^s. Let C be a component of H_0 cut along $\partial_\pm N$ and along any periodic orbits contained in H_0 such that $C \subset N$. If C is not a disk it is an annulus, and it is one of four types (illustrated in Figure 6):*

1. *(compact periodic) C is an expanding or contracting periodic half-leaf of $W_N^{u/s}$ and has a unique closed orbit on its boundary. Its other boundary component is a single closed loop in $\partial_\pm$ to which the flow lines are transverse.*
2. *(noncompact periodic) C is an expanding or contracting periodic half-leaf of $W_N^{u/s}$ and has a unique closed orbit on its boundary. The rest of ∂C is a collection of properly embedded lines of $H_0 \cap \partial_\pm N$, to which the flow lines are transverse.*
3. *(transient compact) ∂C consists of two closed curves where the flow intersects both ∂_+ and ∂_- transversely.*
4. *(transient noncompact) ∂C has one closed component meeting one of $\partial_\pm$, and a collection of arcs meeting the other.*

If C is contained in a blown annulus it has type (1) or (3).

If C is periodic and expanding (contracting), then all flow lines in C are backward (forward) asymptotic to γ and all other boundary components of C lie in $\partial_+ N$ ($\partial_- N$).

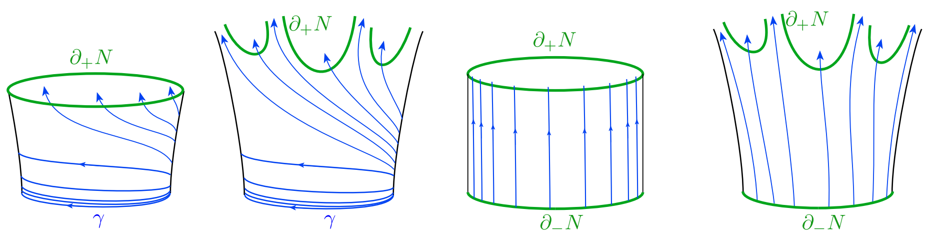

FIGURE 6. The types of annular half-leaves of $W_N^{s/u}$. Cases (1), (2) and (4) are shown in the expanding case.

Proof. We suppose H_0 lies in W^u; the stable case is analogous.

In each blown annulus of H_0, the flow φ is asymptotic to one boundary component in each direction. Cutting along the core curves where the blown annulus intersects $\partial_\pm$, we obtain annuli of types (1) and (3).

Now consider a noncompact half-leaf A of H_0, with ∂A equal to a closed orbit γ. It is a standard fact that each end of a leaf of the stable or unstable foliation of a pseudo-Anosov diffeomorphism of a compact surface is dense in the surface (even after a dynamic blowup), and this implies that A is dense in M. Hence any subset of A whose complement is compact has nonempty intersection with $\partial_\pm$.

On $\mathrm{int}(A)$, the flow is equivalent up to diffeomorphism to the vertical flow on $S^1 \times \mathbb{R}$ (where the backward time flow to $-\infty$ spirals toward γ). The intersection with $\partial_\pm$, by transversality, consists of closed curves or arcs which are graphs of functions from S^1 or a subinterval (respectively) to $\mathbb{R}$. Note this forces the arcs to be properly embedded and asymptotic to the $+\infty$ direction.

From this description we see that every component of the complement of these curves of intersection is either a disk or an annulus, and that each annulus component C either contains γ in its boundary (giving cases (1) and (2)) or is bounded below by a single closed curve of $\partial_\pm$ (giving (3) and (4)).

The proof of the final statement now follows from the reasoning above together with the observation that orbits of the flow enter N only though $\partial_- N$ and exit N only though $\partial_+ N$. $\square$

Next, we want to describe how a periodic half-leaf of $W_N^{s/u}$ intersects the surface L, or equivalently any of the noncompact leaves of the depth one foliation $\mathcal{F}$ of N.

Lemma 4.5. *Let H be a periodic half-leaf of $W_N^{u/s}$. The fibration $\mathrm{int}(N) \to S^1$ determined by $\mathcal{F}$ restricts to a fibration of $\mathrm{int}(H)$, where each fiber is the intersection of $\mathrm{int}(H)$ with one of the leaves of $\mathcal{F}$. Moreover, $L \cap H$ is either:*

1. *(compact periodic) A finite union of rays, each of which is transverse to the periodic boundary γ and spirals onto the opposite boundary component.*
2. *(noncompact periodic) A finite union of rays, each of which is transverse to the periodic boundary γ and accumulates onto all of the other boundary components.*

In particular, the interior of H is the suspension of a periodic half-leaf of the foliation $\mathcal{W}^{s/u}$ in L. When f is expanding on the half-leaf, its suspension is expanding, and when f is contracting its suspension is contracting.

Conversely, every suspension of a periodic half-leaf of $\mathcal{W}^{s/u}$ gives rise to such a periodic half-leaf of $W_N^{u/s}$.

See Figure 7 for the two cases of Lemma 4.5.

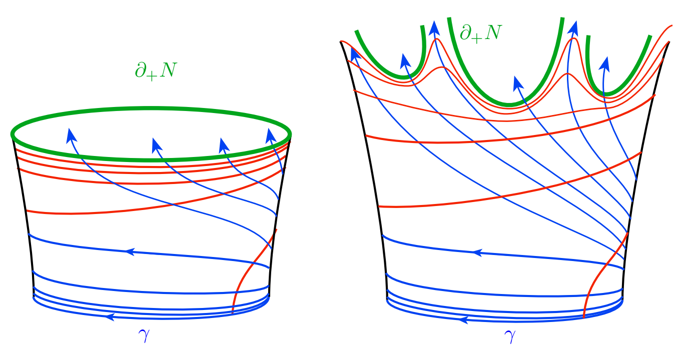

FIGURE 7. Expanding periodic half-leaves of $W_N^{u/s}$ with their intersections with L shown in red.

Proof. The fibration map $\mathrm{int}(N) \to S^1$, restricted to the half-leaf H, is a submersion since the flow directions are transverse to the foliation $\mathcal{F}$. Each leaf F of $\mathcal{F}$ in $\mathrm{int}(N)$ is the preimage of a point in S^1, and so the same is true for $F \cap H$. Finally, $F \cap H$ is a section of the flow in H, because every forward flow ray of φ_N eventually returns to F. This implies that H is fibered over S^1, with fibers $F \cap H$ (which need not be connected).

Since H is a periodic half-leaf, the fibration restricted to the (periodic) boundary component γ is a covering map to S^1, so that $L \cap \gamma$ is a finite union of $k > 0$ points. Thus $L \cap H$ has k components, each of which must be a half-leaf emanating from γ. Let η be one such component. Then $H \cap \mathrm{int}(N)$ is the suspension of f^k restricted to η, so each orbit of $\varphi_N|_{\mathrm{int}(H)}$ returns infinitely often to η. This picture shows that H is expanding/contracting if and only if f^k is expanding/contracting on η.

Suppose H is expanding. In this case $H \cap \partial_- = \varnothing$ by Lemma 4.4. Let ℓ be any component of $H \cap \partial_+$, let $p \in \ell$, and let γ_p be the orbit of φ_N terminating at p. Since $\gamma_p \cap \mathrm{int}(N)$ intersects η infinitely often in the forward direction, we conclude η accumulates on ℓ. The case of H contracting is symmetric.

Conversely, if we suspend a half-leaf, we obtain an annulus which is properly embedded in N and hence must be a periodic annulus component as in Lemma 4.4. □

Proof of Lemma 4.3. By Lemma 4.5, any periodic half-leaf ℓ of $\mathcal{W}^{u/s}$ suspends to a periodic half-leaf H of $W_N^{u/s}$ as in Figure 7. When ℓ is expanding, H is expanding and so ℓ accumulates on ∂_+. In terms of the surface L, this implies that ℓ accumulates on a positive end of L. When ℓ is contracting, H is contracting and ℓ accumulates on ∂_-.

If H is contained in a blown annulus, it corresponds to case (1) of Figure 7 by Lemma 4.4, and in that case the ℓ spirals onto a single boundary component of $\partial_\pm$, which implies that it escapes a unique end. □

4.2. **Hyperbolic metrics, boundaries, and invariant foliations.** We now need to connect the foliations $\mathcal{W}^{u/s}$ of L with its hyperbolic geometry.

A **standard** hyperbolic metric on the surface L is a complete hyperbolic metric that contains no embedded hyperbolic half spaces. Suppose that L is given a standard hyperbolic metric so that its universal cover $\widetilde{L}$ is isometric to the hyperbolic plane; we denote its hyperbolic boundary by $\partial\widetilde{L}$ and the associated compactification by $\mathbb{D} = \widetilde{L} \cup \partial\widetilde{L}$.

The following facts, which we use without further comment, will be crucial:

(1) For any homeomorphism $g\colon L \to L$, any lift $\widetilde{g}\colon \widetilde{L} \to \widetilde{L}$ has a unique continuous extension to $\mathbb{D}$ [CC13, Theorem 2] and we continue to denote by this homeomorphism and its restriction to $\partial\widetilde{L}$ by $\widetilde{g}$.
(2) If $f, g\colon L \to L$ are homotopic maps, then they are isotopic [CC13, Corollary 10] and if $\widetilde{f}, \widetilde{g}\colon \widetilde{L} \to \widetilde{L}$ are obtained by lifting a homotopy to $\widetilde{L}$, then $\widetilde{f}$ and $\widetilde{g}$ agree on $\partial\widetilde{L}$ [CC13, Corollary 5].
(3) If σ_1, σ_2 are two standard hyperbolic metrics on L, then any lift of the identity map on L extends to a homeomorphism $\mathbb{D}_1 \to \mathbb{D}_2$, where $\mathbb{D}_i$ is the hyperbolic compactification of $\widetilde{L}$ with respect to σ_i [CCF19, Lemma 10.1].

The following proposition implies the properties that we will need concerning the singular foliations $\mathcal{W}^{u/s}$, most of which follow from work of Fenley [Fen09]. We let $\widetilde{\mathcal{W}}^{u/s}$ denote the lifts of $\mathcal{W}^{u/s}$ to $\widetilde{L}$.

Proposition 4.6 (Foliations)**.** *Let $f\colon L \to L$ be spA. Then there exists a standard hyperbolic metric on L such that the leaves of $\widetilde{\mathcal{W}}^{u/s}$ are uniformly quasigeodesic.*

Moreover, lifting to the universal cover of M, the intersection of $\widetilde{L}$ with a leaf of $\widetilde{W}^{u/s}$ is connected and hence a leaf of $\widetilde{\mathcal{W}}^{u/s}$.

Proof. First, the moreover statement is precisely [Fen09, Proposition 4.2] and follows easily from the basic structure of $\mathcal{F}$ and φ.

Next, we recall that since M is hyperbolic, Candel proved that M admits a leafwise hyperbolic metric, i.e. a metric which varies continuously and for which each leaf of the foliation $\mathcal{F}$ has constant curvature -1 [Can93].

Now L spirals onto the boundary leaves, and each component of $\partial_+ N$ ($\partial_- N$) contains a closed curve which comes via the semiflow from a closed curve in L (namely, a curve in a fundamental domain of the "endperiodic part" of the map f). By continuity of the metric, the forward (backward) f–orbit of this curve consists of bounded length curves. All points in L are a bounded distance from one of these bounded length curves, so the injectivity radius in the hyperbolic metric on L is bounded above, and in particular the metric is standard.

Finally, let $\varphi^\flat$ be the circular pseudo-Anosov flow on M obtained by blowing down φ. It is well-known that the stable/unstable foliations of $\varphi^\flat$ have Hausdorff leaf space (in fact, they are $\mathbb{R}$-trees); see for example [CD03, Example 2.7]. Since the blowup does not change the leaf space of the stable/unstable foliations, the same is true for $\widetilde{W}^{u/s}$; see the discussion at the end of Section 3 of [Fen09]. Since the foliations $\widetilde{\mathcal{W}}^{u/s}$ are determined by the intersection $\widetilde{L} \cap \widetilde{W}^{u/s}$, the leaf spaces of $\widetilde{\mathcal{W}}^{u/s}$ are also Hausdorff. Hence, we may apply [Fen09, Theorem C] to conclude that the leaves of $\widetilde{\mathcal{W}}^{u/s}$ are uniformly quasigeodesic in $\widetilde{L}$. □

Remark 4.7. If $g\colon L \to L$ is endperiodic and f_1, f_2 are each spA maps isotopic to g, then we can choose a standard hyperbolic metric on L so that, lifting to the universal cover, the leaves of the invariant foliations of *both* f_1 and f_2 are uniformly quasigeodesic.

To see this, let $N = N_g$ be the compactified mapping torus for g and let $N \to M_1$ and $N \to M_2$ be the embeddings associated to f_1 and f_2 as in Theorem 3.3. As in the proof of Proposition 4.6, each M_i has a leafwise hyperbolic metric g_i and these each pull back to a continuous leafwise hyperbolic metric on N, which we also denote g_i. By continuity and compactness of N, the ratio g_1/g_2 is uniformly bounded on N and hence the induced hyperbolic metrics on the leaf L are biLipschitz. In particular, the two metrics on $\widetilde{L}$ are quasi-isometric and so the leaves of both sets of invariant foliations are uniformly quasigeodesic in either metric.

We have the following immediate consequence, which is the primary place where we use that φ is minimally blown up with respect to $\mathcal{F}$.

Lemma 4.8. *Let $f\colon L \to L$ be spA. Any lift $\widetilde{f}\colon \widetilde{L} \to \widetilde{L}$ has at most one fixed point.*

Since powers of spA maps are spA by definition, the same holds for powers of f.

Proof. Suppose $p, q \in \widetilde{L}$ are distinct fixed points of $\widetilde{f}$. Then their respective $\widetilde{\varphi}$–orbits $\widetilde{\gamma}_p$ and $\widetilde{\gamma}_q$ project to homotopic closed φ–orbits γ_p and γ_q in M. If $\gamma_p = \gamma_q$, then N contains an an essential torus, a contradiction. Otherwise, since distinct closed orbits of a circular pseudo-Anosov flow are never homotopic, γ_p and γ_q are closed orbits of a blown leaf λ shared by W^u and W^s. The lifted leaf $\widetilde{\lambda}$ contains $\widetilde{\gamma}_p$ and $\widetilde{\gamma}_q$ and by Proposition 4.6 its intersection $\widetilde{\ell}$ with $\widetilde{L}$ is a (connected) leaf of $\mathcal{W}^{u/s}$ that contains both p and q. Let $\widetilde{a}$ be an arc in $\widetilde{\ell}$ from p to q. Its image in L suspends under φ_N to give an annulus $A \subset \lambda$ contained in the interior of N cobounded by γ_p and γ_q. In particular, A is a union of blown annuli that do not meet the compact leaves of $\mathcal{F}$. This contradicts that φ is minimally blown up with respect to $\mathcal{F}$. □

4.3. **Local dynamics on $\partial\widetilde{L}$.** The first step in proving Theorem 4.1 is to analyze the *local dynamics* of lifts of endperiodic maps to $\widetilde{L}$. This lemma gives conditions for a fixed point on $\partial\widetilde{L}$ to be a sink or a source, not just on the boundary but for points of the disk $\mathbb{D} = \widetilde{L} \cup \partial\widetilde{L}$.

Lemma 4.9. *Let $g\colon L \to L$ be an endperiodic map, fix $n \geqslant 0$, and let $\xi \in \partial\widetilde{L}$ be fixed by a lift $\widetilde{g}^n$ of g^n. Suppose further that there is a quasigeodesic ray $\widetilde{r}$ in $\widetilde{L}$ with $\widetilde{r}_\infty = \xi$ such that its projection r to L accumulates on an attracting (repelling) end of L. Then ξ is a sink (source) for the action of $\widetilde{g}^n$ on $\mathbb{D}$.*

Before the proof we need a definition, following [CCF19]. Let $g\colon L \to L$ be an endperiodic map and let e be an attracting end of L with period q. A **g-juncture** is a compact 1-manifold J which is the boundary of a neighborhood U of an end e of L such that

- $g^q(U) \subset U$
- $\bigcap_{k\geqslant 0} g^{kq}(U) = \varnothing$.

The sequence $(g^k(J))$ consists of disjoint curves and accumulates precisely on the ends $\{e, g(e), \ldots, g^{q-1}(e)\}$. If e is a repelling end, then a g-juncture for e is defined to be a g^{-1}-juncture for e. Note that since L has no boundary, each end of L has a *connected* g-juncture.

Proof. For simplicity, we will rename g^n by g.

Let J be a connected geodesic juncture for the end e. Let $\mathcal{J}$ be the collection of all lifts of J to $\widetilde{L}$.

If q is the period of e, then $(g^{i+qk}(J))_{k\geqslant 0}$ escapes to $g^i(e)$ for $0 \leqslant i \leqslant q-1$. By [CCF19, Theorem 4.24], the same is true if we replace all these curves by their geodesic tightenings. Using a superscript $*$ to denote geodesic tightening, this implies that the sequences

$$(\widetilde{g}^k(\mathcal{J}))_{k\geqslant 0} \text{ and } (\widetilde{g}^k(\mathcal{J})^*)_{k\geqslant 0}$$

both escape compact sets in $\widetilde{L}$.

Since r accumulates on e, the elements of $\bigcup_k \widetilde{g}^k(\mathcal{J})$ that essentially intersect $\widetilde{r}$ (that is, whose geodesic tightenings intersect $\widetilde{r}$) cobound a neighborhood basis for ξ in $\mathbb{D}$. This has the following two implications:

- First, ξ is an isolated fixed point of $\widetilde{g}$: suppose for a contradiction that ξ is accumulated by fixed points of $\widetilde{g}$. Then we can find an element $\widetilde{j} \in \bigcup_k \widetilde{g}^k(\mathcal{J})$ and distinct fixed points $a, b, c \neq \xi$ in $\partial\widetilde{L}$ such that $\mathrm{cl}(\widetilde{j})$ separates $\{a, b\}$ from $\{c, \xi\}$ in $\mathbb{D}$. Since $(\widetilde{g}^k(\mathcal{J}))_{k\geqslant 0}$ escapes compact sets in $\widetilde{L}$, $\widetilde{g}^k(\widetilde{j})$ must accumulate on an interval in $\partial\widetilde{L}$ containing $\{a, b\}$ or $\{c, \xi\}$. This prevents the geodesic tightenings of $(\widetilde{g}^k(\widetilde{j}))_k$ from escaping compact sets in $\widetilde{L}$, a contradiction.
- Second, ξ must be either a source or a sink for the action of $\widetilde{g}$ on $\partial\widetilde{L}$. If ξ were an index 0 fixed point (i.e. attracting on one side and repelling on the other), this would force elements of $\bigcup_k \widetilde{g}^k(\mathcal{J})$ near ξ to intersect, another contradiction.

Given that ξ is a source or a sink in $\partial\widetilde{L}$, there exists some other $\widetilde{g}$-fixed point $\zeta \in \partial\widetilde{L}$. Let γ be the oriented geodesic from ζ to ξ, noting that the projection of γ to L also accumulates on e. At this point, we replace J with a connected geodesic juncture that intersects the projection of γ to L (if J does not do so already). Let U be the neighborhood of e bounded by J, and let $\mathcal{U}$ be the collections of all lifts of U and J to $\widetilde{L}$; each element of $\mathcal{U}$ is bounded by infinitely many elements of $\mathcal{J}$.

Let $\mathcal{U}_\gamma$ be the collection of elements of $\mathcal{U}$ through which γ passes. If $A, B \in \mathcal{U}_\gamma$ and γ passes through A before B, set $A < B$; this endows $\mathcal{U}_\gamma$ with a linear order.

Case 1: *$\mathcal{U}_\gamma$ has a maximal element $\widetilde{U}_{max}$.* In other words, γ is eventually contained in $\widetilde{U}_{max}$. Let $\widetilde{V}$ be the smallest closed half space of $\mathbb{D}$ that contains ξ and $\widetilde{U}_{max}$, but not ζ.

Since $\widetilde{g}(\xi) = \xi$ and each element of $\mathcal{U}_\gamma$ is mapped into another by $\widetilde{g}$, we must have $\widetilde{g}(\widetilde{U}_{max}) \subset \widetilde{U}_{max}$ and hence $\widetilde{g}(\widetilde{V}) \subset \widetilde{V}$, so $\widetilde{g}^{k+1}(\widetilde{V}) \subset \widetilde{g}^k(\widetilde{V})$ for $k \geqslant 0$. Since $\widetilde{g}^k(\widetilde{V})$ contains ξ and is bounded by a lift of $g^k(J)$, and $(\widetilde{g}^k(\mathcal{J})^*)_{k\geqslant 0}$ escapes compact sets in $\widetilde{L}$, we must have that

$$\bigcap_{k\geqslant 0} \widetilde{g}^k(\widetilde{V}) = \xi,$$

so ξ is a sink for $\widetilde{g}$ in $\mathbb{D}$.

Case 2: *$\mathcal{U}_\gamma$ has no maximal element.* We again claim that ξ is a sink for $\tilde{g}$ in $\partial\tilde{L}$. Suppose by way of contradiction that it is a source.

Choose a lift $\widetilde{U}_0$ that is not the minimal element of $\mathcal{U}_\gamma$ (it is possible that no minimal element exists). Hence, $\widetilde{U}_0$ does not contain a neighborhood of ζ and so we can let $\widetilde{V}_0$ be smallest closed half space in $\mathbb{D}$ containing ξ and $\widetilde{U}_0$, but not ζ. Finally, let $R \subset \partial L$ be a small repelling neighborhood of ξ so that its convex hull $\widehat{R} \subset \mathbb{D}$ is contained in $\widetilde{V}_0$.

Since the projection of γ accumulates on e and is not eventually contained in U, the set $\widetilde{g}^k(\mathcal{U}_\gamma)$ (ordered also by its intersections with γ) also has no maximal element for all k. In particular $\widetilde{g}^k(\mathcal{U}_\gamma) \cap \widetilde{V}_0$ is nonempty for all k, and hence $(\widetilde{g}^k(\mathcal{J})^*)$ intersects the compact set $\widetilde{V}_0 \cap \gamma \cap \mathrm{cl}(\mathbb{D} - \widehat{R})$ nontrivially for arbitrarily large k, a contradiction.

Hence ξ is a sink for $\widetilde{g}$ in $\partial\widetilde{L}$. We can choose an element $\widetilde{j} \in \mathcal{J}$ close enough to ξ so that the ideal endpoints of $(\widetilde{g}^k(\widetilde{j}))$ converge to ξ. Since the sequence $(\widetilde{g}^k(\widetilde{j}))$ escapes compact subsets of $\widetilde{L}$, this shows ξ is a sink for $\widetilde{g}$ in $\mathbb{D}$. □

We will primarily use Lemma 4.9 in the following form:

Corollary 4.10. *Let $f : L \to L$ be spA and let r be an expanding (contracting) half-leaf in L of period n. Let $\widetilde{r}$ be a lift of r to $\widetilde{L}$. Let $\widetilde{r}_\infty$ denote the endpoint of $\widetilde{r}$ on $\partial\widetilde{L}$, and let $\widetilde{f}^n$ be a lift of f^n to $\widetilde{L}$ such that $\widetilde{f}^n(\widetilde{r}) = \widetilde{r}$. Then*

- *(a) $\widetilde{r}_\infty$ is a sink (source) for the action of $\widetilde{f}^n$ on $\mathbb{D}$, and*
- *(b) if $g\colon L \to L$ is an endperiodic map homotopic to f, and $\widetilde{g}^n$ is a lift of g compatible with the lift $\widetilde{f}^n$, then $\widetilde{r}_\infty$ is also a sink (source) for the action of $\widetilde{g}^n$ on $\mathbb{D}$.*

Proof. We equip L with a standard hyperbolic metric so that the leaves of $\widetilde{\mathcal{W}}^{u/s}$ are uniformly quasigeodesic (Proposition 4.6). By Lemma 4.3, the expanding (contracting) half-leaf $\widetilde{r}$ accumulates on a positive (negative) end of L. Now apply Lemma 4.9 to both f and g. □

4.4. **Global dynamics on the hyperbolic boundary.** We are now ready to prove one direction of Theorem 4.1, namely that the dynamics of half-leaves at fixed points of an spA map determine the corresponding dynamics at infinity. The other direction will be proved in Proposition 4.15.

Proposition 4.11. *Let $f\colon L \to L$ be spA with fixed point p. Let $\widetilde{p}$ be a lift of p to $\widetilde{L}$, let $\widetilde{f}$ be a lift that fixes $\widetilde{p}$, and let $n \geqslant 1$ be such that $\widetilde{f}^n$ fixes the half-leaves at $\widetilde{p}$. Then $\widetilde{f}^n$ has multi sink-source dynamics on $\partial\widetilde{L}$ with attracting/repelling points equal to the endpoints of expanding/contracting half-leaves.*

Before giving the proof, we will require the following lemma. First, following Fenley, we say that a **slice leaf** of a singular foliation is a line that is the union of two half-leaves meeting at their common initial point. The slice leaf is a **leaf line** if the two half-leaves are adjacent in the circular ordering of half-leaves about their common initial point. Note that regular leaves are leaf lines and any nonregular slice leaves contain a singularity.

Lemma 4.12. *Let $\widetilde{\mathcal{W}}$ be either $\widetilde{\mathcal{W}}^u$ or $\widetilde{\mathcal{W}}^s$ and suppose that $x, y \in \partial\widetilde{L}$ are not separated by any leaf of $\widetilde{\mathcal{W}}$. Then there is a leaf $\ell \in \widetilde{\mathcal{W}}$ with $\{x, y\} \subset \partial\ell$.*

Proof. Let A be one of the two closed intervals in $\partial\widetilde{L}$ with endpoints x, y.

Define a partial order on the leaf lines of $\widetilde{\mathcal{W}}$ with endpoints in A as follows: $\ell_1 \prec_A \ell_2$ if the endpoints of ℓ_2 determine a subinterval of A containing the subinterval determined by the endpoints of ℓ_1. Using Proposition 4.6, it is clear that every linear chain has an upper bound and so there is a maximal element ℓ_A by Zorn's lemma. We claim that ℓ_A must join x to y.

Let p be any point of ℓ_A and p_i a sequence converging to p such that each p_i is contained in the complementary component of the closure of ℓ_A in $\mathbb{D}$ that meets A^c. Let ℓ_i be a leaf line of $\widetilde{\mathcal{W}}$ through p_i. Then the ℓ_i converge to a leaf line ℓ' through p. Note that the endpoints of ℓ' are not in the interior of A. Indeed, no endpoints of ℓ_i are contained in A because this would contradict either the maximality of ℓ_A or the assumption that no leaf of $\widetilde{\mathcal{W}}$ separates x and y.

We conclude that there exists a (possibly singular) leaf ℓ of $\widetilde{\mathcal{W}}$ containing ℓ_A and ℓ'. Suppose toward a contradiction that ℓ_A has an endpoint c in the interior of A. If ℓ' has an endpoint outside A, then ℓ contains a slice leaf separating x and y, a contradiction. On the other hand if ℓ' joins x and y then this contradicts the maximality of ℓ_A. We conclude that ℓ_A has no endpoint in the interior of A, so $\partial\ell_A = \{x, y\}$. ☐

Proof of Proposition 4.11. After replacing f by f^n, we may assume that f fixes the half-leaves at p. We remark that if one half-leaf at p is fixed, then they all are.

Let r be an expanding half-leaf at $\widetilde{p}$. We will show that that if r' is a contracting half-leaf that is adjacent to r in the cyclic order around $\widetilde{p}$, then each point in the innermost interval $(r'_\infty, r_\infty) \subset \partial\widetilde{L}$ is attracted to r_∞ under positive powers of $\widetilde{f}$ and to r'_∞ under negative powers of $\widetilde{f}$. The case where r is a contracting half-leaf is symmetric and so this will complete the proof.

There are two cases:

Case 1: *r (or r') is not a blown half-leaf.*

Note this is the case if p is nonsingular. Assume that r is not a blown half-leaf; the case for r' is similar.

Since r is expanding and not a blown half-leaf, it is a half-leaf of $\widetilde{\mathcal{W}}^u$ and its interior is transverse to $\widetilde{\mathcal{W}}^s$. Let r', r'' be the contracting half-leaves starting at $\widetilde{p}$ that are adjacent to r in the cyclic ordering (when p is not singular, these are the only half-leaves of $\widetilde{\mathcal{W}}^s$ through $\widetilde{p}$), and let ℓ be any leaf line of $\widetilde{\mathcal{W}}^s$ that crosses the interior of r. We show that the endpoints $\widetilde{f}^k(\partial\ell)$ converge to r_∞ as $k \to \infty$ and to the two point set $\{r'_\infty, r''_\infty\}$ as $k \to -\infty$. We recall that by Proposition 4.6 the leaves of $\widetilde{\mathcal{W}}^{u/s}$ are uniformly quasigeodesic.

First, as $k \to -\infty$, the leaves $\widetilde{f}^k(\ell)$ meet r along points that converge to $\widetilde{p}$. Since the interiors of r' and r'' do not meet singularities of $\widetilde{\mathcal{W}}^s$, $f^k(\ell)$ must converge to the leaf line

$r' \cup r''$ as $k \to -\infty$ since this is the unique leaf line of $\widetilde{\mathcal{W}}^s$ through $\widetilde{p}$ that meets the side of $r' \cup r''$ containing r. Hence, $\widetilde{f}^k(\partial\ell)$ converges to $\{r'_\infty, r''_\infty\}$ as $k \to -\infty$ as claimed.

Next, as $k \to \infty$, the leaves $\widetilde{f}^k(\ell)$ meet r along points that exit r and hence converge to r_∞. If $\widetilde{f}^k(\ell)$ exits compact sets of $\widetilde{L}$, then the sequence of leaves converges to r_∞ and hence $\widetilde{f}^k(\partial\ell)$ converge to r_∞ as required. Otherwise, as $k \to \infty$, $\widetilde{f}^k(\ell)$ converges to a line in a leaf $\widehat{\ell}$ of $\widetilde{\mathcal{W}}^s$ which must have r_∞ as an endpoint. Let ℓ' be the leaf of $\widetilde{\mathcal{W}}^s$ containing $\widetilde{p}$ (and hence containing r' and r'' as half-leaves).

Hence, $\widehat{\ell}$ and ℓ' are distinct leaves of $\widetilde{\mathcal{W}}^s$ that are invariant under $\widetilde{f}$. But this contradicts the fact that the lift of an spA map fixes at most one leaf of $\widetilde{\mathcal{W}}^s$ (or $\widetilde{\mathcal{W}}^u$), which we show as follows: The lift $\widetilde{f}$ extends to a deck translation τ of $\widetilde{N}$, and since $\pi_1 N \to \pi_1 M$ is injective, τ extends to a deck translation of $\widetilde{M}$. Now ℓ' and $\widehat{\ell}$ suspend to give leaves R and E, respectively, of the stable foliation $\widetilde{W}^s$ of $\widetilde{\varphi}$ in $\widetilde{M}$ which are fixed under τ, and this implies that their closed orbits are homotopic. Since distinct homotopic closed orbits of φ in M are contained in the same (necessarily blown) leaf of W^s, we must have that $R = E$, and by Proposition 4.6 we have that the intersection $\widetilde{L} \cap R$ is connected. But since it contains both ℓ' and $\widehat{\ell}$, this is a contradiction.

Case 2: *Both r and r' are blown half-leaves.*

In this case, r and r' are half-leaves of both $\widetilde{\mathcal{W}}^s$ and $\widetilde{\mathcal{W}}^u$. Let $[r_\infty, r'_\infty] \subset \partial\widetilde{L}$ be the interval between their endpoints that does not contain the other endpoints of half-leaves based at $\widetilde{p}$. By Corollary 4.10, on the action of $\widetilde{f}$ on $[r_\infty, r'_\infty]$ is such that r_∞ is locally an attractor and r'_∞ is locally a repeller. Hence, it suffices to show that there are no other fixed points in $[r_\infty, r'_\infty]$.

Suppose towards a contradiction that (r_∞, r'_∞) contains a fixed point of $\widetilde{f}$. We will show that there is an $\widetilde{f}$–invariant leaf line $\widehat{\ell}$ with endpoints in $[r_\infty, r'_\infty)$, which gives a contradiction exactly as at the end of Case 1 (since r and r' are already fixed by $\widetilde{f}$, and cannot be in the same leaf with $\widehat{\ell}$).

Let q_1 be the fixed point closest to r_∞; this exists since the fixed point set of $\widetilde{f}$ is closed and any point sufficiently close to r_∞ cannot be fixed. First suppose that there is a leaf line ℓ of $\widetilde{\mathcal{W}}^s$ that separates r_∞ from q_1. Then the limit $\widetilde{f}^k(\ell)$ as $k \to \infty$ is an $\widetilde{f}$–invariant leaf line $\widehat{\ell}$ of $\widetilde{\mathcal{W}}^s$ that joins r_∞ with some fixed point q in (r_∞, r'_∞) (here we are using the fact that r'_∞ is a local repeller). If no such separating leaf line exists, then by Lemma 4.12 we obtain a leaf line ℓ of $\widetilde{\mathcal{W}}^s$ with endpoints r_∞ and q_1. If ℓ is not fixed by $\widetilde{f}$, consider the family of all leaf lines joining r_∞ to q_1. This set is closed and $\widetilde{f}$–invariant and so its boundary lines are fixed, and we let $\widehat{\ell}$ be one of them. The contradiction now proceeds as in Case 1. $\square$

As a consequence of Corollary 4.10 and Proposition 4.11 we can show that two isotopic spA maps have the same local dynamics. This will be essential in Section 4.5.

Lemma 4.13. *Suppose that $f, f' \colon L \to L$ are homotopic spA maps. Let $\widetilde{f}, \widetilde{f}'$ be corresponding lifts to $\widetilde{L}$ and suppose that $\widetilde{f}$ fixes a point $\widetilde{p}$ and its half-leaves. Then f' also fixes a (unique) point $\widetilde{q}$ and its half-leaves. There is a bijective correspondence between the half-leaves at $\widetilde{p}$ and $\widetilde{q}$ induced by having the same limit point on $\partial\widetilde{L}$.*

Moreover, a half-leaf $\widetilde{r}$ at $\widetilde{p}$ has image in L that escapes to an end e if and only if the corresponding leaf $\widetilde{r}'$ at $\widetilde{q}$ has the same property.

The proof will use the following well-known fact. The proof we give appears in Farb–Margalit [FM12], where it is attributed to Handel [Han85].

Lemma 4.14. *Let $g\colon \mathbb{D} \to \mathbb{D}$ be a homeomorphism with at least 4 fixed points on $\partial\mathbb{D}$ each of which has an attracting or repelling neighborhood in $\mathbb{D}$. Then g has a fixed point in* $\mathrm{int}(\mathbb{D})$.

Proof. Double the map to get a homeomorphism $Dg\colon S^2 \to S^2$. By assumption, this map has at least 4 fixed points along the equator, each of which is attracting or repelling and hence has positive Lefschetz index. Since $\chi(S^2) = 2$, there must be a fixed point of negative index. Since this fixed point necessarily occurs off the equator, we have found a fixed point of g in $\mathrm{int}(\mathbb{D})$. □

Proof of Lemma 4.13. By part (a) of Corollary 4.10, each endpoint of each half-leaf at $\widetilde{p}$ has an attracting/repelling neighborhood in $\mathbb{D}$. By part (b), the same is true for the action of $\widetilde{f}'$. Hence, by Lemma 4.14, $\widetilde{f}'$ has a fixed point $\widetilde{q}$ in $\widetilde{L}$. We then apply Proposition 4.11 to $\widetilde{f}'$ at $\widetilde{q}$ to complete the proof of the first claim.

For the moreover statement, we apply Remark 4.7 to choose a standard metric on L so that the leaves of both sets of invariant foliations on L have uniformly quasigeodesic lifts to $\widetilde{L}$. Hence, $\widetilde{r}$ and $\widetilde{r}'$ fellow travel in $\widetilde{L}$ and so their images fellow travel in L. In particular, one escapes e if and only if the other does. This completes the proof. □

4.5. **Existence of spA$^+$ representatives.** We will now prove Theorem 4.2 (Theorem B from the introduction) which states that any atoroidal endperiodic map is in fact isotopic to an spA$^+$ map.

Proof of Theorem 4.2. The proof proceeds just as for Theorem 3.3, where we must now show that for an appropriate $M = M(h, N)$, we have $\varphi^\flat = \varphi$, i.e. $\partial_\pm N$ is positively transverse to the circular pseudo-Anosov flow φ. We assume, as we may, that N is not a product.

First choose an auxiliary spA map f_a as in Theorem 3.3. Using the spA package associated to f_a, we can consider the collection C_1 of isotopy classes of curves in $\partial_\pm N$ that are closed leaves of the singular foliation $\partial_\pm N \cap W_a^{s/u}$ on $\partial_\pm N$. Also, let C_2 be the collection of curves in $\partial_\pm N$ that cobound essential annuli of N from $\partial_- N$ to $\partial_+ N$. Let $C = C_1 \cup C_2$.

Now let $h\colon \partial_\pm N \to \partial_\pm N$ be a component-wise homeomorphism that satisfies Proposition 3.9 and for which no curve of C is mapped into C, up to isotopy. This can be achieved just as in the proof of Lemma 3.12. Let $M = M(h, N)$ be the associated h–double (Section 3.2). As in Lemma 3.12, M is hyperbolic and we consider the spA map $f^\dagger$ produced in the proof of Theorem 3.3.

We show that $f^\dagger$ is spA$^+$. Referring to the proof of Theorem 3.3, it suffices to show that φ has no blown annuli, i.e. that $\varphi^\flat = \varphi$. Suppose otherwise that there is a blown annulus A. After cutting A along $\partial_\pm$ we obtain, as in Lemma 4.4 and Lemma 4.5, annuli corresponding to cases (1) and (3) of Figure 6. Case (3) yields an essential annulus in N, which intersects $\partial_\pm$ in curves of C_2. Case (1) yields the suspension of an escaping (Lemma 4.3), expanding/contracting half-leaf of $f^\dagger$. We show this latter case meets $\partial_\pm$ in curves of C_1. Note this is not immediate since $W^{u/s}$ and $W_a^{u/s}$ are foliations on different manifolds with different flows.

Let r be an escaping, expanding/contracting half-leaf of $f^\dagger$. By Lemma 4.13, there is a corresponding escaping, periodic half-leaf r_a of f_a. Moreover, from a spiraling neighborhood U of $\partial_\pm N$, one sees that the suspensions H and H_a of r and r_a under their respective flows produce homotopic curves $H \cap \partial_\pm N$ and $H_a \cap \partial_\pm N$. Hence, we have show that all curves of $\partial_\pm N \cap A$ are contained in C. But from the construction of $M = M(h, N)$, we see that h maps curves in $\partial_+ N \cap A$ to curves in $\partial_- N \cap A$ and hence curves in C back into C. This contradicts our choice of h and completes the proof. □

4.6. **Sink-source dynamics imply fixed points.** With Theorem 4.2 in hand we can complete the proof of Theorem 4.1. The remaining direction is stated in this proposition:

Proposition 4.15. *Let $f\colon L \to L$ be spA. Suppose that $\widetilde{f}$ is a lift to $\widetilde{L}$ such that $\widetilde{f}^n$ acts with multi sink-source dynamics on $\partial\widetilde{L}$ for some $n \geqslant 1$. Then $\widetilde{f}$ has a unique fixed point $\widetilde{p}$ in $\widetilde{L}$ whose half-leaves are fixed by $\widetilde{f}^n$ such that the endpoints of expanding/contracting half-leaves at $\widetilde{p}$ are exactly its attracting/repelling points in $\partial\widetilde{L}$.*

Proof. It suffices to show the existence of the (necessarily unique by Lemma 4.8) fixed point $\widetilde{p}$ since the rest then follows from Proposition 4.11. Note that we cannot directly use the Lefschetz argument of Lemma 4.14 as before, because our hypothesis does not give sink/source properties in the disk $\mathbb{D}$, only on its boundary.

Further, if $\widetilde{p}$ is fixed by $\widetilde{f}^n$, then it must also be fixed by $\widetilde{f}$ since otherwise $\widetilde{f}^n$ has multiple fixed points. Hence, it suffices to replace $\widetilde{f}$ with $\widetilde{f}^n$ and we do so now.

Next, by Lemma 4.13 we are free to replace f with any isotopic spA map; let $f^\dagger$ be an isotopic spA$^+$ map whose existence is guaranteed by Theorem 4.2. So it remains to prove that if a lift $\widetilde{f}^\dagger$ has multi sink-source dynamics on $\partial\widetilde{L}$ then it has a fixed point in $\widetilde{L}$. The key place we use that $f^\dagger$ is spA$^+$ is that its invariant foliations $\mathcal{W}^{u/s}$ are transverse away from singularities, i.e. there are no blown leaves.

Let x_1, x_2 be attracting fixed points of $\widetilde{f}^\dagger$ on $\partial\widetilde{L}$. First suppose that they are joined by slice leaf ℓ of $\mathcal{W}$, where $\mathcal{W}$ denotes either $\mathcal{W}^u$ or $\mathcal{W}^s$ (and let $\mathcal{W}'$ denote the other one). As in the proof of Proposition 4.11, by choosing ℓ to be a boundary leaf of the family of all lines of $\mathcal{W}$ joining x_1, x_2 we can assume that $\widetilde{f}^\dagger(\ell) = \ell$. Let ℓ' be any regular leaf of, say, $\mathcal{W}'$ that crosses ℓ. By the multi sink-source dynamics, the end points of $(\widetilde{f}^\dagger)^{-k}(\ell')$ converge to distinct repelling points y_1, y_2 in $\partial\widetilde{L}$, as $k \to \infty$. We conclude that there is an $\widetilde{f}^\dagger$-invariant leaf line ℓ'' joining y_1, y_2 and the intersection $\ell \cap \ell''$ in $\widetilde{L}$ is the required fixed point $\widetilde{p}$.

Now suppose x_1 and x_2 are not joined by a slice leaf of either $\mathcal{W}^{u/s}$. Then according to Lemma 4.12 they are separated by a slice leaf ℓ of $\mathcal{W}^u$ (or of $\mathcal{W}^s$). Because there are infinitely many choices for ℓ we can assume it has endpoints that are not fixed. Again considering the limit of $(\widetilde{f}^\dagger)^{-k}(\ell)$ as $k \to \infty$, the endpoints converge to distinct repelling points $y_1, y_2 \in \partial\widetilde{L}$, which are therefore joined by a leaf of one of the foliations. This then reduces to the previous case, with $\widetilde{f}^\dagger$ replaced by $(\widetilde{f}^\dagger)^{-1}$. □

5. Invariant laminations and topological entropy

Over the next two sections we establish the various characterizations of the stretch factor of an spA map as given in Theorem C. Here, we focus on the topological entropy of spA maps, whereas the next section is devoted to the sense in which spA maps are dynamically optimal.

For this, recall from Section 1.2 that for any homeomorphism $g\colon L \to L$, its **growth rate** $\lambda(g)$ is the exponential growth rate of its periodic points:

$$\lambda(g) = \limsup_{n\to\infty} \sqrt[n]{\#\mathrm{Fix}(g^n)},$$

where $\mathrm{Fix}(g)$ denotes the set of fixed points of g. Given an spA map $f\colon L \to L$, there is a largest compact invariant *core dynamical system* $C_f \subset L$ defined in Section 5.2, and the entropy $\mathrm{ent}(f)$ is defined to be the topological entropy of the restriction of f to C_f. The main theorem of this section is:

Theorem 5.1. *If f is spA, then* $\mathrm{ent}(f) = \log \lambda(f)$.

Along the way, we define canonical invariant laminations $\Lambda^\pm$ associated to an spA map f (Theorem 5.3) and show that C_f depends only on the homotopy class of the spA map: two homotopic spA maps have conjugate core dynamical systems (Proposition 5.15).

5.1. **Invariant laminations.** Let $f\colon L \to L$ be an spA map. A point $x \in L$ is called **positive escaping** if $f^n(x)$ exits compact sets through positive ends of L as $n \to \infty$ and called **negative escaping** if $f^n(x)$ exits compact sets through negative ends of L as $n \to -\infty$.

Definition 5.2. *Let Λ^+ be the union of all points in L which are not negative escaping, and let Λ^- be the union of all points in L which are not positive escaping.*

We will call Λ^+ and Λ^- the **positive** and **negative invariant laminations** for f, respectively. We will justify this terminology by proving in Theorem 5.3 that they are the supports of (singular) sublaminations of $\mathcal{W}^u$ and $\mathcal{W}^s$, respectively. In general, a *sublamination* Λ of a singular foliation $\mathcal{W}$ is a closed subset that is the union of subleaves of $\mathcal{W}$, where a *subleaf* of $\mathcal{W}$ is the union of at least 2 half-leaves. Note that each leaf is itself a subleaf and a subleaf not containing a singularity of $\mathcal{W}$ is an entire (regular) leaf. See Proposition 5.8 for the precise description of how $\Lambda^\pm$ are obtained from $\mathcal{W}^{u/s}$.

We can now state the main theorem of this subsection.

Theorem 5.3 (Invariant laminations)**.** *For an spA map $f\colon L \to L$, the invariant laminations $\Lambda^\pm$ are a pair of f–invariant, transverse, nowhere dense (singular) laminations such that:*

1. *each periodic expanding half-leaf of $\mathcal{W}^u$ (resp. periodic contracting half-leaf of $\mathcal{W}^s$) is a half-leaf of Λ^+ (resp. Λ^-),*
2. *the collection of periodic leaves of $\Lambda^\pm$ is dense in $\Lambda^\pm$,*
3. *each half-leaf of Λ^+ (resp. Λ^-) accumulates on a positive (resp. negative) end of L but not on a negative (resp. positive) end of L.*

The proof of Theorem 5.3 will appear in Section 5.1.2.

We will observe in Remark 5.17 that the laminations $\Lambda^\pm$ depend only on the isotopy class of f rather than the specific choice of spA representative.

5.1.1. *Laminations in N.* For the proof of Theorem 5.3, we return to the compactified mapping torus N of f and its semiflow φ_N. We refer the reader to Section 4.2 for notation surrounding the invariant foliations $W^{u/s}_N = W^{u/s} \cap N$.

Let W^+_N be the union of all φ_N-orbits which do not meet $\partial_- N$, and let W^-_N be the union of all φ_N-orbits which do not meet $\partial_+ N$. For the next lemma, a **periodic blown half-leaf** of $W^{u/s}_N$ is the suspension of a periodic blown half-leaf of $\mathcal{W}^{u/s}$. In other words, it is a component of the intersection of a leaf of $W^{u/s}_N$ with a blown annulus in N that contains a closed orbit of φ_N. See Figure 5.

Lemma 5.4. *Let $W^\pm_N$ and $W^{u/s}_N$ be as above.*

(a) Let H be a leaf of W^u_N. Then one of the following is true:
- *H does not contain a periodic blown half-leaf, in which case H meets $\partial_- N$ if and only if the backward orbit of every point of H meets $\partial_- N$, or*
- *H contains at least one periodic blown half-leaf, in which case the points in H whose backward orbits meet $\partial_- N$ are exactly those points in the interior of contracting blown half-leaves.*

(b) The set W^+_N is obtained from W^u_N by removing
- *every leaf H with the property that for all $p \in H$, the backward φ_N-orbit from p meets $\partial_- N$, and*
- *the open periodic blown half-leaves which are contracting.*

Statements (a) and (b) are true after replacing 'W^u_N' with 'W^s_N,' 'backward with 'forward,' '$\partial_- N$' with '$\partial_+ N$,' and 'contracting' with 'expanding'.

Proof. Note that (a) implies (b), so we will simply prove (a). If H is periodic, this follows immediately from Lemma 4.4, where each periodic half-leaf of H is either of type (1) or (2).

Now suppose that H is not periodic and consider the flow φ on M with its unstable foliation W^u. Since H is a leaf of W^u_N that is *not* periodic, it does not contain a closed orbit of a blown annulus. As the closed orbits in the boundary of a blown annulus do not intersects $\mathcal{F}_0$ (again see Figure 5), this implies that H does not intersect any closed orbit of a blown annulus. Hence all backwards φ-orbits of points in H are asymptotic *in M*. The following claim implies that a single backward orbit in H meets $\partial_- N$ if and only if they all do.

Claim 5.5. *Suppose φ is a flow in a compact 3-manifold M with universal cover $\widetilde{M}$, that $\widetilde{\varphi}$ is the lift of φ to $\widetilde{M}$, and that Σ is a surface positively transverse to φ. Suppose $\widetilde{x}, \widetilde{y} \in \widetilde{M}$ both lie above a fixed lift $\widetilde{\Sigma}$ of Σ. If the backward $\widetilde{\varphi}$-orbits from $\widetilde{x}$ and $\widetilde{y}$ are asymptotic, then $\widetilde{x}$ intersects $\widetilde{\Sigma}$ in backward time under $\widetilde{\varphi}$ if and only if $\widetilde{y}$ does.*

The same holds after replacing 'above' with 'below,' and 'backward' with 'forward.'

Proof of Claim 5.5. Let $\widetilde{N}_\varepsilon$ be a regular neighborhood of $\widetilde{\Sigma}$ obtained by lifting a regular ε-neighborhood of Σ, where ε is small enough so that $\widetilde{x}, \widetilde{y} \notin \widetilde{N}_\varepsilon$ and the foliation of $\widetilde{N}_\varepsilon$ by flowlines gives $\widetilde{N}_\varepsilon$ the structure of an oriented interval bundle over $\widetilde{\Sigma}$.

Let $\widetilde{\gamma}_x$ and $\widetilde{\gamma}_y$ be the the backward orbits from $\widetilde{x}$ and $\widetilde{y}$ respectively. Suppose that $\widetilde{\gamma}_x$ meets $\widetilde{\Sigma}$. Since N_ε is an I-bundle, $\widetilde{\gamma}_x$ eventually lies below $\widetilde{\Sigma}$ and outside of $\widetilde{N}_\varepsilon$. Since the backward orbits from $\widetilde{x}$ and $\widetilde{y}$ are asymptotic, $\widetilde{\gamma}_y$ must also eventually lie below $\widetilde{\Sigma}$. Since the roles of $\widetilde{x}$ and $\widetilde{y}$ are symmetric, we are done. □

Returning to our leaf H, let $p, q \in H$ and suppose the backward orbit from p meets $\partial_- N$. let $\widetilde{H}$ be a lift of H to $\widetilde{M}$, and let $\widetilde{p}, \widetilde{q} \in \widetilde{H}$ be lifts of p and q respectively. The backward $\widetilde{\varphi}$-orbit from $\widetilde{p}$ intersects some lift $\widetilde{\Sigma}$ of a component Σ of $\partial_- N$. Since the backward orbits from $\widetilde{p}$ and $\widetilde{q}$ are asymptotic, they must both intersect $\widetilde{\Sigma}$ by Claim 5.5. Projecting to M, we see that the backward orbit from q must intersect $\partial_- N$.

The case where H is a leaf of W^s_N follows symmetrically, finishing the proof of Lemma 5.4. □

Proposition 5.6 (Laminations $W^\pm_N$)**.** *The pair $W^\pm_N$ are nowhere dense (singular) sublaminations of $W^{u/s}_N$ such that each boundary leaf contains a closed orbit of φ.*

In the proof, we will use the flow space of φ, defined as follows. Let $\widetilde{M}$ denote the universal cover of M. Collapsing orbits of the lifted flow defines a quotient map $\Theta\colon \widetilde{M} \to \mathcal{O}$, and $\mathcal{O}$ is called the **flow space** of φ. According to Fenley–Mosher [FM01, Proposition 4.1, 4.2], $\mathcal{O}$ is homeomorphic to the plane. Note that the images of $W^{u/s}$ in $\mathcal{O}$ are a pair of singular foliations.

Proof of Proposition 5.6. We will focus on W_N^+ since the case for W_N^- is similar.

To show that W_N^+ is a sublamination of W_N^u, note that its complement is the set of points whose backward φ_N-orbits meet $\partial_- N$. This set is clearly open, so W_N^+ is closed. By Lemma 5.4(b), W_N^+ is a union of subleaves of W_N^u, and hence is a (singular) sublamination. Here, we have used that fact that every singular orbit of φ_N bounds at least 2 contracting and 2 expanding half-leaves.

That W^+ is nowhere dense follows from the fact φ has an orbit in M that is dense in both the forward and backward direction. This orbit exists because every pseudo-Anosov flow on an atoroidal manifold is transitive [Mos92, Proposition 2.7].

It remains to prove that each boundary leaf of W_N^+ contains a closed orbit of φ. For this, let H be a boundary leaf of W_N^+. If we let H_M be the leaf of W^u that contains H, we claim that it suffices to show that H_M contains a closed orbit of φ. Indeed, for any point $p \in H \subset H_M \cap N$, the backward φ-orbit from p limits on a closed orbit of H_M. However, since H is a leaf of W_N^+ the backwards orbit through p never meets $\partial_- N$ and hence stays in N. In particular, H contains a closed orbit of H_M.

To prove that H_M contains a closed orbit, we use the projection $\Theta\colon \widetilde{M} \to \mathcal{O}$ from the universal cover of $\widetilde{M}$ to the flow space $\mathcal{O}$ of φ. First, let $\widetilde{N}$ be a component of the preimage of N in $\widetilde{M}$ and let $\widetilde{H}$ be a component of the preimage of H contained in N. Since H is a boundary leaf, there is a component Σ of $\partial_- N$ and a lift $\widetilde{\Sigma}$ to the boundary of $\widetilde{N}$ with the property that points arbitrarily close to $\widetilde{H}$ flow backwards to meet $\widetilde{\Sigma}$. Hence, the image $\Theta(\widetilde{H})$ lies at the boundary of the $\Theta(\widetilde{\Sigma})$. (For context, the topological boundary of $\Theta(\widetilde{\Sigma})$ is a union of slices leaves of the projected foliations [Fen09, Theorem 4.1].) But we prove in [LMT24, Proposition 8.7] that each slice leaf in the boundary of $\Theta(\widetilde{\Sigma})$ is periodic. (For φ an honest pseudo-Anosov suspension flow, this was proved in [CLR94, Lemma 3.21].) This implies that H_M contains a periodic orbit as required. $\square$

5.1.2. *Proof of Theorem 5.3.* It remains to translate properties of $W_N^\pm$ into properties of the laminations $\Lambda^\pm$.

Corollary 5.7. *Let ℓ be a leaf of $\mathcal{W}^u$. Then one of the following is true:*

- *ℓ does not contain a periodic blown half-leaf, in which case ℓ contains a negative escaping point if and only if every point in ℓ is negative escaping, or*
- *ℓ contains at least one periodic blown half-leaf, in which case the negative escaping points in ℓ are exactly those contained in the interiors of contracting blown half-leaves.*

The same hold for leaves of $\mathcal{W}^s$ after replacing 'negative' with 'positive' and 'contracting' with 'expanding.'

Proof. Since $f\colon L \to L$ is the first return map to L under the flow φ and the positive (negative) ends of L accumulate precisely on $\partial_+ N$ ($\partial_- N$), we have $\Lambda^\pm = W_N^\pm \cap L$. Moreover, leaves of $W_N^\pm$ are in bijective correspondence with f–orbits of leaves of $\Lambda^\pm$ and so the corollary follows from Lemma 5.4. $\square$

Corollary 5.7 and the definition of $\Lambda^{\pm}$ now imply the following proposition:

Proposition 5.8. *The positive invariant lamination Λ^+ is obtained from the foliation $\mathcal{W}^u$ by removing*

- *the leaves for which every point is negative escaping, and*
- *the open periodic (blown) half-leaves that are contracting.*

The negative invariant lamination Λ^- is characterized symmetrically by replacing '$\mathcal{W}^u$' with '$\mathcal{W}^s$,' 'negative' with 'positive,' and 'contracting' with 'expanding.'

Proof of Theorem 5.3. First, since $W_N^{\pm}$ are nowhere dense sublaminations of $W_N^{u/s}$ by Proposition 5.6, $\Lambda^{\pm} = W_N^{\pm} \cap L$ are nowhere dense sublaminations of $\mathcal{W}^{u/s}$.

Let ℓ be an expanding periodic half-leaf of $\mathcal{W}^u$. By Lemma 4.3, ℓ does not accumulate on negative ends. Hence, ℓ is not negative escaping and therefore is a half-leaf of Λ^+. Since every periodic point of L is the endpoint of at least two expanding periodic half-leaves of $\mathcal{W}^u$, the same is true in Λ^+; in other words, there are no "1-pronged singular leaves" of Λ^+. The case where ℓ is contracting periodic leaf of $\mathcal{W}^u$ is similar and thus item (1) is proven.

Next we prove item (3). Let r be a half-leaf of Λ^+. Note that r does not accumulate on a negative ends of L, because it contains no negative escaping points. Suppose for a contradiction that r does not accumulate on positive end. Then r is contained in a compact set $K_1 \subset L$, and hence accumulates on a sublamination L_1 of Λ^+. Since $L_1 \subset \Lambda^+$, $f^n(L_1)$ stays in a fixed compact set K_2 as $n \to -\infty$ and so it limits to a compact f-invariant sublamination $L_2 \subset \Lambda^+$. The suspension of L_2 under φ is contained in interior of $N \subset M$, contradicting the fact that all half-leaves of $W^{u/s}$ are dense in M (as in the proof of Lemma 4.3).

It remains to prove that periodic leaves of $\Lambda^{\pm}$ are dense. Since $\Lambda^{\pm}$ are nowhere dense, boundary leaves are dense in each and so it suffices to know that each boundary leaf is f–periodic. Let ℓ be a boundary leaf of (say) Λ^+. Then the suspension of ℓ under φ_N is a boundary leaf H of W_N^+. According to Proposition 5.6, H contains a periodic orbit γ and so its intersection $\gamma \cap L$ includes an f–periodic point in ℓ. This gives item (2), completing the proof. ☐

Remark 5.9 (Periodic leaves of $\Lambda^{\pm}$)**.** From the argument above, we see that f–periodic leaves of $\Lambda^{\pm}$ contain f–periodic points. That is, if $f^n(\ell) = \ell$ for ℓ a leaf of $\Lambda^{\pm}$, then ℓ contains a (unique) fixed point of f^n. Indeed, the suspension of ℓ under the semiflow is a leaf H of $W_N^{\pm}$ that is contained in a periodic leaf H_M of $W^{u/s}$. As in the proof of Proposition 5.6, if H is a leaf of W_N^+ (resp. W_N^-) then following a point of H backwards (resp. forwards) along the flow, we stay in $H \subset N$ and arrive at a periodic orbit of φ. Intersecting this orbit with ℓ produces the required f–periodic point.

Remark 5.10 (Invariant measures on $\Lambda^{\pm}$)**.** An example of Fenley [Fen97], together with Theorem 8.4, shows that $\Lambda^{\pm}$ do not always admit projectively invariant transverse measures of full support. It would be interesting to characterize when there are such transverse measures.

5.2. **Topological entropy of spA maps.** We next associate to each spA map $f\colon L \to L$ its **core dynamical system** C_f:

$$C_f = \{x \in L : (f^n(x))_{n\in\mathbb{Z}} \text{ is bounded in } L\}.$$

Since f is endperiodic, C_f is exactly the set of points that are neither positive nor negative escaping. In symbols, $C_f = \Lambda^+ \cap \Lambda^-$.

Lemma 5.11. *The core C_f is f–invariant, compact, and contains any other f–invariant compact set.*

We define the **entropy** of an spA map f, denoted $\mathrm{ent}(f)$, to be the topological entropy of the restriction $f|_{C_f}: C_f \to C_f$. For background on topological entropy in the related setting of pseudo-Anosov maps, see [FLP79, Exposé 10].

Remark 5.12. Topological entropy is usually (and unambigously) defined for maps on compact spaces, but there have been several generalizations. For example, Cánovas and Rodríguez define the topological entropy of a general map to be the supremum of topological entropies of the restriction to compact invariant subset [CR05]. With this definition, $\mathrm{ent}(f)$ is unchanged by Lemma 5.11.

An endperiodic map on L is a **translation** if every point of L is both positive and negative escaping. The following lemma characterizes translations among spA maps.

Lemma 5.13. *Let $f: L \to L$ be spA. The following are equivalent:*

(1) one of Λ^+ or Λ^- is empty,
(2) both Λ^+ and Λ^- are empty,
(3) f has no periodic points,
(4) C_f is empty,
(5) f is a translation.

Proof. Any periodic point has a bounded orbit, so (4) implies (3). Each periodic point is contained in a leaf of both Λ^+ and Λ^-, and by Theorem 5.3 the periodic leaves of $\Lambda^\pm$ are dense in $\Lambda^\pm$, so (3) implies (2). Clearly (2) implies (1), and (1) implies (4) since $C_f = \Lambda^+ \cap \Lambda^-$ as noted above. Hence (1) through (4) are equivalent.

To finish the proof, we note that (2) and (5) are equivalent because $\Lambda^+ = \varnothing$ and $\Lambda^- = \varnothing$ if and only if each point of L is both positive and negative escaping, i.e. f is a translation. □

With an eye toward proving Theorem 5.1, we recall the fact, due to Bowen [Bow71, Theorem 17], that if $S: X \to X$ and $T: Y \to Y$ are maps and $F: X \to Y$ is continuous, surjective, equivariant, and has finite fibers, then the topological entropies of S and T are equal. In particular, conjugate systems have the same topological entropy.

The next proposition shows that the core dynamical system of an spA map is naturally conjugate to the restriction of a pseudo-Anosov homeomorphism to a closed invariant subspace.

Proposition 5.14. *Suppose $f: L \to L$ is an spA map with associated closed manifold M and circular flow φ (Remark 3.5) so that there is a cross section S of φ obtained by de-spinning (Remark 3.14). If $F: S \to S$ is the first return map for S, then $f: C_f \to C_f$ is conjugate to the restriction of F to some closed, invariant subspace.*

Proof. Since S is obtained by de-spinning, there is an isotopy along flow lines carrying $S \cap N$ to a compact subsurface L' of L, such that the isotopy is supported in a spiraling neighborhood U of $\partial_\pm N$ (see Section 3.1). In the notation of Section 3, $S \cap N = L_U$ and $L' = L \setminus U$. Since U is foliated by flow segments, the isotopy from $L' \subset L$ to $S \cap N \subset S$ along segments of the flow is supported away from C_f. With this setup, $C_f \subset S$, and f agrees with F since they are both the return map to C_f along φ. □

Next we show that the core dynamical system does not depend on the choice of spA representative. In other words, it is canonically associated to the isotopy class of f.

Proposition 5.15. *If $f_1, f_2: L \to L$ are isotopic spA maps, then their core dynamical systems are conjugate.*

For the proof, we require a bit more structure associated to the core. First, we have the following lemma about the endpoints of the lifts $\widetilde{\Lambda}^\pm$ of $\Lambda^\pm$ to $\widetilde{L}$:

Lemma 5.16. *Each slice leaf of $\widetilde{\Lambda}^\pm \subset \widetilde{L}$ is uniquely determined by its endpoints in $\partial\widetilde{L}$, and leaves of $\widetilde{\Lambda}^+$ and $\widetilde{\Lambda}^-$ do not share endpoints.*

Moreover, if ℓ_1 and ℓ_2 are leaves of (say) Λ^+ that share an endpoint at infinity, then they share a periodic half-leaf based at a singularity.

Proof. We begin by proving the moreover statement. Suppose that ℓ_1 and ℓ_2 are distinct leaves of $\widetilde{\Lambda}^+$ that share an endpoint $p \in \partial\widetilde{L}$. It suffices to show that they share a half-leaf since distinct leaves that share a half-leaf are singular and all singularities in Λ^+ are periodic: if s is a singularity of Λ^+ then $f^n(s)$ stays in a compact set of L as $n \to -\infty$ and hence becomes periodic. If ℓ_1 and ℓ_2 do not share a half-leaf, we obtain a contradiction as follows: since the laminations Λ^+ are nowhere dense, there is also a boundary leaf line ℓ of $\widetilde{\Lambda}^+$ with endpoint p. Since boundary leaves are periodic (Proposition 5.6), p itself is fixed under some $\widetilde{f}^k$. If both ℓ_1, ℓ_2 are also fixed under $\widetilde{f}^k$, then we are done since they both contain the unique fixed point of $\widetilde{f}^k$. If not, then iterating either of ℓ_1, ℓ_2 under negative powers of $\widetilde{f}^k$ limits to a leaf line of $\widetilde{\Lambda}^+$ that joins an attracting point to a repelling point by Theorem 4.1. But this produces a ray in a leaf line of $\widetilde{\Lambda}^+$ with the same endpoint as a ray in a leaf line of $\widetilde{\Lambda}^-$. Projecting these rays to L, we obtain a contradiction to the third item of Theorem 5.3.

The remaining claims now follow easily. If ℓ_1 and ℓ_2 are slice leaves of $\widetilde{\Lambda}^+$ with the same endpoints, then $\ell_1 = \ell_2$ because they agree on two singular half-leaves and each leaf contains at most one (necessarily periodic) singularity (Lemma 4.8). Finally, if a leaf ℓ^+ of $\widetilde{\Lambda}$ shared an endpoint with a leaf ℓ^- of $\widetilde{\Lambda}$, then their projection to L would fellow travel. But this again contradicts item (3) of Theorem 5.3. ☐

Next, recall that the **double boundary** $\partial^2\widetilde{L}$ is defined as the space of distinct pairs of points in the circle $\partial\widetilde{L}$ modulo the involution $(x, y) \mapsto (y, x)$. The laminations $\Lambda^\pm$ determine closed, pairwise unlinked, $\pi_1(L)$–invariant subsets $\partial^2(\Lambda^\pm) \subset \partial^2\widetilde{L}$ obtained by taking the endpoints of *leaf lines* (see Section 4.4) in $\widetilde{\Lambda}^\pm \subset \widetilde{L}$. In light of Lemma 5.16, we are free to blur the distinction between a leaf line of $\widetilde{\Lambda}$ and the corresponding point in $\partial^2(\Lambda^\pm)$. Since $\Lambda^\pm$ are f-invariant, $\partial^2(\Lambda^\pm)$ are invariant under the homeomorphism induced by any lift of $\widetilde{f}$ of f, which we continue to denote by $\widetilde{f}$.

Define
$$\widetilde{\Lambda}_\pitchfork = \{(l^+, l^-) \in \partial^2(\Lambda^+) \times \partial^2(\Lambda^-) : l^+ \text{ and } l^- \text{ link in } \partial\widetilde{L}\},$$
and note that $\widetilde{\Lambda}_\pitchfork$ is $\pi_1(L)$ invariant and $\widetilde{f}(\widetilde{\Lambda}_\pitchfork) = \widetilde{\Lambda}_\pitchfork$ for each lift of f. In fact, $\widetilde{\Lambda}_\pitchfork$ admits a natural equivariant surjection to the preimage $\widetilde{C}_f$ of the core in $\widetilde{L}$. To see this, define the map $\widetilde{\Lambda}_\pitchfork \to \widetilde{C}_f$ that sends each (l^+, l^-) to the unique point of intersection $l^+ \cap l^-$. This map is surjective essentially by definition, and working in foliation charts of $\widetilde{\mathcal{W}}^\pm$, we see that the map is also continuous. Hence, by equivariance, it descends to a continuous surjective map that conjugates the induced action of f on $\Lambda_\pitchfork = \widetilde{\Lambda}_\pitchfork/\pi_1(L)$ to the action of f on C_f.

In fact, the fibers of the map $\Lambda_\pitchfork \to C_f$ are easily determined (see e.g. Casson–Bleiler [CB88, Lemma 6.2]). In particular, once sees that the fibers of the map $\Lambda_\pitchfork \to C_f$ generate an equivalence relation $\sim$ that is entirely determined by $\Lambda_\pitchfork$ and that the the fibers have uniformly bounded size.

With this setup, we give the

Proof of Proposition 5.15. From the above discussion, each spA map f_i determines a subspace $\partial^2(\Lambda_i^\pm) \subset \partial^2\widetilde{L}$ in which the periodic leaf lines are dense. However, by Lemma 4.13, the collection of periodic leaf lines of these laminations are equal and hence $\partial^2(\Lambda^\pm) := \partial^2(\Lambda_1^\pm) = \partial^2(\Lambda_2^\pm)$. Since the maps f_1, f_2 are homotopic, compatible lifts have the same action on $\partial\widetilde{L}$ and so f_1, f_2 have the same action on $\Lambda_\pitchfork := \Lambda_{1\pitchfork} = \Lambda_{2\pitchfork}$. Finally, the factor maps $\Lambda_\pitchfork \to C_{f_1}$ and $\Lambda_\pitchfork \to C_{f_2}$ make exactly the same identifications by above discussion. Hence, the cores C_{f_1} and C_{f_2} are conjugate, as required. □

Remark 5.17 (Invariance of laminations)**.** We record the fact, which follows from the proof of Proposition 5.15, that if f_1 and f_2 are isotopic spA maps, then their respective invariant laminations $\Lambda_1^\pm$ and $\Lambda_2^\pm$ are isotopic (they have the same geodesic tightenings and they differ from their tightenings by a controlled pinching).

We now use Proposition 5.14, Proposition 5.15, and the symbolic dynamics of pseudo-Anosov homeomorphisms to give the

Proof of Theorem 5.1. According to Proposition 5.15, it suffices to prove the theorem for any isotopic spA map and by Theorem 4.2 we may choose f to be spA$^+$. That is, if we consider the spA package associated with f, as in Remark 3.5, then the flow φ is an honest pseudo-Anosov suspension flow on M. We further assume that M is constructed as an h–double, as in the proof of Theorem 4.2, so that the foliation $\mathcal{F}$ can be de-spun to a cross section S of φ; see Remark 3.14. In particular, the first return map $F\colon S \to S$ produced by Proposition 5.14 is an honest pseudo-Anosov homeomorphism. Let C be the closed F–invariant subspace of S that is conjugate to C_f; it suffices to show that the entropy of $F|_C$ is equal to the logarithm of the growth rate of its fixed points.

Let $\sigma\colon \Sigma \to \Sigma$ be a subshift of finite type associated to a Markov partition for $F\colon S \to S$ as constructed in [FLP79, Exposé 10]. The symbolic coding $\theta\colon \Sigma \to S$ is a semiconjugacy with the property that that its fibers have uniformly bounded size. Hence, the same is true for the restricted coding: $\theta|_{\theta^{-1}(C)}\colon \theta^{-1}(C) \to C$. Since the entropy of a subshift of finite type is given by the logarithm of the growth rate of its fixed points [Bow70], it suffices to show that $\theta^{-1}(C) \subset \Sigma$ has finite type.

According to Fried [Fri87, p. 492], it suffices to show that the subsystem $C \subset S$ is *isolated*, i.e. it has an open neighborhood U such that $C = \bigcap_{i\in\mathbb{Z}} F^i(U)$. This is easy from our topological setup: First, reparameterize the flow φ so that $F\colon S \to S$ is the time 1 return map and chose any metric on M. Next, note that $C \subset S$ is exactly the F–invariant closed set with the property that no flow line through its points ever intersects $\Sigma = \partial_\pm N$ (here we are referring to the notation of Proposition 5.14 and Remark 3.5). The φ–transverse surface Σ has an ε–collar foliated by segments of the flow and we choose an open neighborhood $U \subset S$ of C so that for each $u \in U$ there is a $c \in C$ having the property that $d(\varphi_t(u), \varphi_t(c)) \leqslant \varepsilon/2$ for $t \in [-1, 1]$. To show $C = \bigcap_{i\in\mathbb{Z}} F^i(U)$ note that any $x \in \bigcap_{i\in\mathbb{Z}} F^i(U)$ has the property that $d(\varphi_t(x), \varphi_t(C)) \leqslant \varepsilon/2$ for all $t \in \mathbb{R}$. But since $\varphi_t(C) \subset N$ for all t, $\varphi_t(C)$ never meets the ε–collar of Σ (since such points necessarily escape N). Hence, $\varphi_t(u)$ never meets Σ and hence $u \in C$ as required. This completes the proof. □

6. Stretch factors of spun pseudo-Anosov maps

In this section we complete the proof of Theorem C by establishing that spA maps minimize growth rate among all homotopic endperiodic maps (Corollary 6.2) and that for an spA map f, $\lambda(f)$ is equal to the growth rate of intersection numbers of curves under iteration (Theorem 6.4).

Taken together, this is our justification for calling $\lambda(f)$ the **stretch factor** of the spA map $f\colon L \to L$.

6.1. **Growth rates of fixed points.** Having developed the required structure in Section 4, the proof of the following theorem can proceed just as in the case of a pseudo-Anosov homeomorphism. See, for example, [FM12, Theorem 14.20].

Theorem 6.1. *Let $f\colon L \to L$ be a spun pseudo-Anosov map. Then for each $n \geqslant 1$, the homeomorphism f has the minimum number of periodic points of period n among all homotopic, endperiodic homeomorphisms.*

Proof sketch. Let g be another endperiodic map that is homotopic to f. The proof works by showing that for any period n periodic point x for f there is a *Nielsen equivalent* periodic point y of g with the same period n. Here x and y are Nielsen equivalent if there are lifts $\widetilde{x}$ and $\widetilde{y}$ to $\widetilde{L}$ such that compatible lifts of f^n and g^n fix $\widetilde{x}$ and $\widetilde{y}$ respectively. Note that no two periodic points of f are Nielsen equivalent since each lift of $\widetilde{f}^k$ fixes a most one point by Lemma 4.8. Hence, from this fact it follows that Nielsen equivalence determines an injective map from period n periodic points of f to period n periodic points of g.

To this end, suppose that x is a periodic point of f of period n. To simplify notation, replace f (and g) with f^n (and g^n, respectively) so that x is now a fixed point. Let $\widetilde{x}$ be a lift of x and let $\widetilde{f}$ be a lift of f that fixes $\widetilde{x}$. By Proposition 4.11, there is a $k \geqslant 1$ such that $\widetilde{f}^k$ acts with multi sink-source dynamics on ∂L; hence, so does the kth power of a compatible lift $\widetilde{g}$ of g. First suppose that $k = 1$. Then by Lemma 4.9, each fixed point of $\widetilde{g}$ on ∂L is attracting/repelling for the action on $\mathbb{D}$. So by Lemma 4.14, $\widetilde{g}$ has a fixed point $\widetilde{y}$ in $\widetilde{L}$. If $k \geqslant 2$, then we note that $\widetilde{g}$ does not fix any points on ∂L. Then, referring to the proof of Lemma 4.14, the homotopic maps $D\widetilde{g}$ and $D\widetilde{f}$ have no fixed points on the equator and the same Lefschetz index. But the unique fixed point of $\widetilde{f}$ has negative index and so $D\widetilde{g}$, and hence $\widetilde{g}$, must have a fixed point $\widetilde{y}$ in L.

The proof that x and y have the same period now follows exactly as in [FM12, Theorem 14.20]. This completes the proof. ∎

The following corollary is now immediate:

Corollary 6.2. *If $f\colon L \to L$ is spA, then*

$$\lambda(f) = \inf_{g \simeq f} \lambda(g),$$

where the infimum is over all endperiodic maps homotopic to f.

In particular, $\lambda(f)$ for an spA map f depends only on the homotopy class of f.

Remark 6.3. Ellis Buckminster has proved that the term 'endperiodic' can be removed from the statement of Corollary 6.2, thereby answering a question that we asked in an earlier version of this paper [Buc25].

6.2. **Stretch factors and growth of intersection numbers.** Next, we turn to the following purely topological characterization of the stretch factor.

Theorem 6.4. *Let f be a spun pseudo-Anosov map. Then*

$$\lambda(f) = \max_{\alpha,\beta} \limsup_{n\to\infty} \sqrt[n]{i(\beta, f^n(\alpha))}, \tag{1}$$

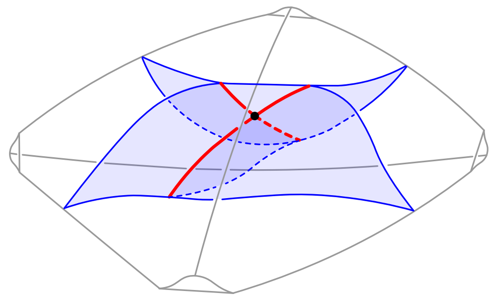

FIGURE 8. Part of the unstable branched surface B^u lying in a single τ-tetrahedron. The branch locus is indicated in red.

where the maximum is taken over all isotopy classes of essential simple closed curves.

Recall that if α and β are simple closed curves in L, then $i(\alpha, \beta)$ denotes their geometric intersection number, i.e. the minimal number of intersection points as α and β are varied within their isotopy class. In particular, it is immediate that the right hand side of eq. (1) depends only on the homotopy class of f.

Remark 6.5. We will prove Theorem 6.4 using an invariant train track $\mathcal{V}$ for $f: L \to L$ (see Section 6.2.3). In the finite-type setting, the analogous intersection property for pseudo-Anosov maps can be proven using the singular flat metric associated to the invariant *measured* foliations. While in our setting this structure does not necessarily exist (see Remark 5.10), there could still be an interesting approach using similar ideas.

The proof of Theorem 6.4 will occupy the next several subsections. We will assume, as we may by Theorem 4.2 and Corollary 6.2, that $f: L \to L$ is an spA$^+$ map as defined in Definition 3.4. Referring to the spA package from Remark 3.5, the foliation $\mathcal{F}$, which has L as a depth one leaf, is transverse to the pseudo-Anosov flow φ on M. We also assume, as in the construction (see Remark 3.11), that $\mathcal{F}$ is obtained by spinning a cross section S of φ about the transverse surface $\Sigma = \partial_\pm N$. Finally, we assume that there exists at least one periodic point of f and hence $\lambda(f) \geqslant 1$; otherwise the equality in Theorem 6.4 holds by Lemma 5.13.

6.2.1. *Some veering combinatorics.* For the purposes of proving Theorem 6.4 we will need some combinatorial tools.

Recall that associated to φ is the veering triangulation τ of the manifold $\mathring{M}$ obtained by removing the singular orbits of φ from M [Ago11, Gué16]. The 2–skeleton of τ is a branched surface in $\mathring{M}$ transverse to φ ([LMT21, Theorem 5.1]) and, after replacing $\mathring{M}$ with the complement of a tubular neighborhood U of the singular orbits, we obtain the 'compact model' of τ. See [Lan22, LMT24] for details.

Also associated to τ is an **unstable branched surface** which we denote B^u (see [LMT20, Section 5.2.1] or [Lan22, Section 3.3]). This branched surface is characterized by the property that it is topologically dual to τ, and its intersection with each τ-face F is a train track whose single switch points toward the unique edge of F which is topmost for the τ-tetrahedron immediately below F. As a consequence, the portion of B^u lying in a single τ-tetrahedron looks like the suspension of a single train track folding move. See Figure 8 where the one-skeleton of B^u is indicated in red; note that there are two branch lines crossing in a single 'triple point' in the center of the tetrahedron.

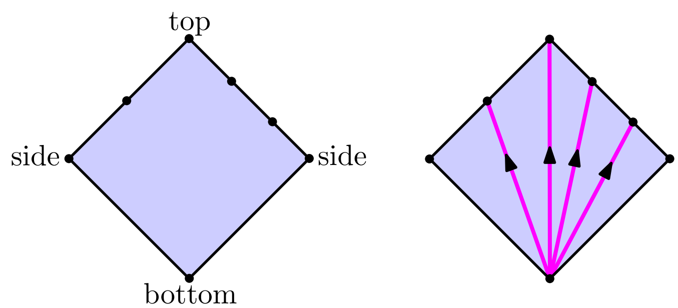

FIGURE 9. Left: a sector of B^u with side, top, and bottom points labeled. Right: the corresponding part of the flow graph Φ in dual position.

We now describe some combinatorics of B^u that was developed in [LMT20, Section 4]. Strictly speaking the combinatorics were developed for a related branched surface B^s, but one can pass between the branched surfaces by reversing the coorientation on τ-faces.

The sectors of B^u are diffeomorphic to rectangles. Let s be a B^u-sector. Then the sides of s inherit orientations from the **dual graph** Γ of τ; this is the directed graph dual to the faces of τ, which is naturally identified with the 1–skeleton of B^u. There is a unique corner of s which is a source, which we call the **bottom point** of s, and a unique sink which we call the **top point** of s. The two remaining corners are called **side points**. The side points are linked in ∂s with the top and bottom points. The complementary components of the two side points containing the top and bottom points of s are the **top** and **bottom** of s, respectively. The bottom point is the unique triple point in the bottom of s, while the top of s may contain any number of triple points. See the lefthand side of Figure 9.

The complementary regions of B^u in M are "cusped solid tori." That is, each is diffeomorphic to the mapping torus of an n-cusped disk by a diffeomorphism. The number n is equal to the number of prongs of the singular φ-orbit at the core of the complementary region, so in particular $n \geqslant 3$.

The final object associated to τ is the **(unstable) flow graph** Φ, which can be defined as follows: its vertices are the edges of τ, and for each τ-tetrahedron there is a directed Φ-edge to the top τ-edge from the bottom τ-edge and the two equatorial τ-edges that are involved with the associated train track folding move (see, e.g., Figure 10). When Φ is embedded in M in this way we say it is in **standard position**. There is another position for Φ we will use called **dual position**. In dual position there is a vertex at each triple point of B^u (i.e. vertex of Γ), and for each B^u-sector s there is a directed edge in s from the bottom triple point of s to each triple point which is not a side vertex. See the righthand side of Figure 9. In dual position, each Φ-edge is positively transverse to the 2-skeleton $\tau^{(2)}$.

Starting with Φ in standard position, we can put Φ in dual position by homotoping it downward with respect to τ [LMT20, Section 4].

6.2.2. *Carried position and the veering track on $\partial_+ N$.* Since Σ is transverse to the pseudo-Anosov flow φ, [LMT24, Theorem E] proves that Σ is 'relatively carried' by $\tau^{(2)}$ in the following way: after an isotopy, Σ intersects the tubular neighborhood U in meridional disks transverse to φ and otherwise is contained in an I-fibered neighborhood of $\tau^{(2)}$ where it is transverse to the fibers.

The intersection of Σ with the unstable branched surface B^u is the (unstable) **veering train track** $\mathcal{V}_\Sigma$ on Σ. The complementary regions $\Sigma \setminus\setminus \mathcal{V}$ are cusped n-gons with $n \geqslant 3$ because of the structure of the complementary regions of B^u discussed earlier.

The carried position of Σ is not unique and as in [Lan19, Proposition 4.5] we can choose Σ in a 'lowest' position. In such a position the track $\mathcal{V}_\Sigma$ has no *large branches*, which are branches whose switches both point inward—a large branch indicates a diagonal exchange that moves Σ to a lower position. We fix Σ once and for all in this position.

Since Σ and S are both carried by τ, we can spin S about Σ to produce the foliation $\mathcal{F}$ whose leaves are carried by τ in the above sense. In particular, L is carried by τ so we can define an associated unstable track $\mathcal{V}$ on L as we did for Σ.

Claim 6.6. *Any essential curve c in Σ can be isotoped to be transverse to $\mathcal{V}_\Sigma$, such that there are no bigons in $\Sigma \setminus\setminus (\mathcal{V}_\Sigma \cup c)$.*

Proof of Claim 6.6. We sketch a proof of the claim for any train track T whose complementary regions are n-gons ($n \geqslant 3$) and which has no large branches. Such a train track carries finitely many closed curves which we will call the *sinks* of T, and a unique geodesic lamination λ with one closed leaf for each sink of T. The sinks are disjoint in T by the no large branches condition. Each end of a noncompact leaf of λ spirals on a closed leaf, and each closed leaf has noncompact leaves spiraling onto it from both sides in a consistent direction. We may isotope T so that its sinks agree with the closed leaves of λ. Note that by spinning the sinks of T in the direction of the spiraling by an isotopy supported in a small neighborhood of the sinks, we can approximate λ arbitrarily well by T.

Given any simple closed curve c which is not isotopic to a sink, its geodesic representative c^* is transverse to λ and all the complementary regions of $c^* \cup \lambda$ have nonpositive index. If c is isotopic to a sink then we can push c^* off the sink slightly in one direction to achieve the same end. Now by a spinning isotopy as described above, we may assume each complementary region of $c^* \cup T$ corresponds to a complementary region of $c^* \cup \lambda$ with the same index, ruling out bigons. ☐

Remark 6.7. Since L is obtained by spinning, the track $\mathcal{V}$ in the ends of L semi-covers the track $\mathcal{V}_\Sigma$. If c is any curve far enough into the ends of L, we can project to Σ, apply Claim 6.6, and lift back to L. Hence, c can be isotoped to be transverse to $\mathcal{V}$ with no complementary bigons.

Cutting the unstable flow graph along Σ. Let Φ be the unstable flow graph of τ embedded in M in dual position.

Let η be the cohomology class dual to Σ. Let $\Phi \setminus\setminus \Sigma$ denote Φ minus the edges which (in our embedding) cross faces of τ that carry Σ with positive weight. Then let $\Phi|\eta$ be the *dynamical core* of $\Phi \setminus\setminus \Sigma$, namely the subset consisting of edges that are contained in directed cycles. We note that $\Phi|\eta$ carries all directed cycles of Φ whose pairing with η is 0.

Let Φ_N be the subgraph of $\Phi|\eta$ which is contained in N.

6.2.3. *The invariant veering track and growth rates.* The argument will hinge on the following constructions, which we will justify below.

(1) The map $f\colon L \to L$ determines a train track folding map $f_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{V}$ by pushing L through the tetrahedra in $\mathrm{int}(N) \setminus\setminus L$ and noting that each step is a folding move on the track, until the final track is mapped back to $\mathcal{V}$ by f.
(2) Let G_f be the transition graph of the folding map $f_{\mathcal{V}}$. That is, the vertices of G_f are the branches of $\mathcal{V}$, and for each branch b, the map $f_{\mathcal{V}}$ takes it to a sequence

of branches and G_f contains a directed edge from b to c for each appearance of a branch c in this sequence. When $f_{\mathcal{V}}$ maps b homeomorphically to a single branch c, we get just one edge emanating from b, which terminates in c, and we call that a simple edge.

(3) G_f is equipped with an embedding $h_1 : G_f \to N$.

(4) h_1 is homotopic to a map $h_2 : G_f \to \Phi$ which is "monotonic" in the sense that each edge of G_f is either collapsed to a vertex of Φ or mapped in an orientation-preserving manner to a directed path in Φ.

(5) h_2 gives a bijection between positive cycles in G_f and positive cycles in Φ_N, so that the number of edges in a cycle in G_f is equal to the intersection number of its image with L.

Once we have G_f, we define the growth rate $\mathrm{gr}_{G_f}(1)$ as the exponential growth rate of directed cycles of length n, where length is just number of edges (i.e each edge has length one). Our main claim about this is

Claim 6.8. $\lambda(f) = \mathrm{gr}_{G_f}(1)$.

We will prove this claim after describing points (1–5) of the construction.

We first describe the folding map $f_{\mathcal{V}}$. Because M is fibered with fiber S, there is some multiple mS which is fully carried by the 2-skeleton $\tau^{(2)}$. Since L is obtained by spinning S around Σ, we obtain (possibly enlarging m) a cycle of parallel copies $L = L_0, L_1, \ldots, L_m = L$ of L, each carried by τ, so that between two successive copies there is exactly one tetrahedron. The passage from L_i to L_{i+1} (mod m) applies a folding move to the train track $\mathcal{V}_i = L_i \cap B^u$. Composing these we obtain the map $f_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$. This gives part (1).

Part (2) gives the definition of G_f.

We obtain the map $h_1 : G_f \to N$ of part (3) as follows:

Realize the union of the L_i in carried position in a fibered regular neighborhood $\mathcal{N}$ of the 2-skeleton of τ in $\mathring{M}$. For clarity we work in the punctured manifold, and so each leaf L_i is punctured by the singular orbits and admits an ideal triangulation: For each cell c (face or edge) of the veering triangulation let $\mathcal{N}(c)$ be the union of fibers of $\mathcal{N}$ passing through c. We then get an ideal triangulation of L_i whose cells are components of the intersection of L with each $\mathcal{N}(c)$.

Between L_i and L_{i+1}, there is one tetrahedron t_i so that L_i passes along its bottom faces and L_{i+1} passes along the top. The folding move is illustrated in Figure 10. We can build an intermediate graph G_i connecting the triangulation edges in L_i to those in L_{i+1}. Outside of t_i, the directed edges of G_i are mapped along vertical fibers of $\mathcal{N}$ from a triangulation edge of L_i to the parallel copy above it. Within the tetrahedron we have such vertical edges together with additional edges indicating the folding, as in the figure.

Now each edge of G_f corresponds to a directed path in the union of the G_i, from L_0 back to itself. This gives the map $h_1 \colon G_f \to N$ described in part (3).

Now we observe that the three edges terminating in e' in Figure 10 correspond to the edges of the flow graph Φ for that tetrahedron, using the definition of Φ in standard position. Each of the vertical G_i edges in the figure (as well as any outside this tetrahedron) is contained in a product region $\mathcal{N}(e)$ for some edge e. Thus each edge of the intermediate graphs G_i can either be mapped to an edge of Φ or collapsed to a vertex of Φ. Following the map h_1 from edges of G_f to paths along the G_i with this map to Φ, we obtain the map h_2. Note that, in the dual position embedding of Φ, each vertex moves down from its corresponding

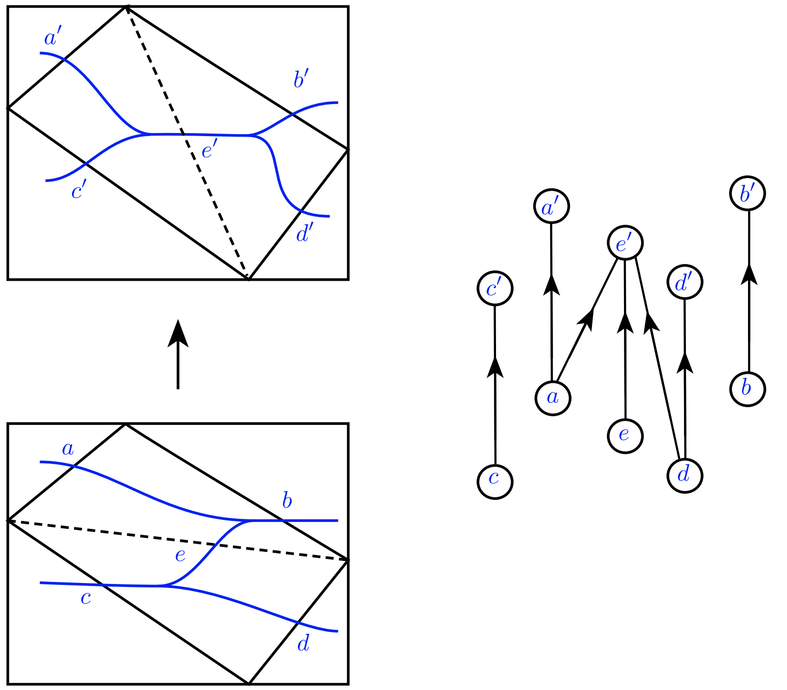

FIGURE 10. A folding map across a single tetrahedron produces edges in G_f

τ edge to the interior of the tetrahedron below it. This is a homotopy, so we see that the final map h_2 is homotopic to h_1. This establishes part (4).

It remains to explain item (5), the correspondence between cycles of G_f and cycles of Φ. By the construction of h_2, every directed cycle of G_f maps to a directed cycle of Φ. Moreover, because h_2 and h_1 are homotopic, and h_1 maps to the complement of Σ, the intersection number of $h_2(c)$ with Σ is 0 for each cycle c of G_f. Since Φ is positively transverse to Σ this implies that $h_2(c)$ lies in $\Phi|\eta$. Finally each cycle in G_f crosses L non-trivially which places the image cycle in Φ_N.

Conversely given a directed cycle c in Φ_N we must find a cycle in G_f that maps to it. Note that c must have positive intersection with L because N is fibered by L. We will obtain the cycle in G_f by cutting c along its intersections with L, but first we consider the local picture.

Let r be an edge of Φ_N that c traverses. Then r begins in the interior of a tetrahedron t_- whose top edge e_- is identified with the initial vertex of r. It passes transversely through the 2-skeleton of τ to end in a tetrahedron t_+, for which e_- is either a bottom or side edge. After a slight homotopy we can arrange for r to pass in t_- directly through a small neighborhood U of e_- and from there into t_+. See Figure 11.

Note that Σ, in its carried position, is not intersected by r since r is an edge of Φ_N. But r might intersect a part of L passing through U, and this must happen for at least one such r through which c travels. Thus, r cut along its intersections with L gives us a sequence of segments in U that are homotopic within U to simple edges of the graph G_f (in its h_1 embedding), followed by one edge that we identify with the original r.

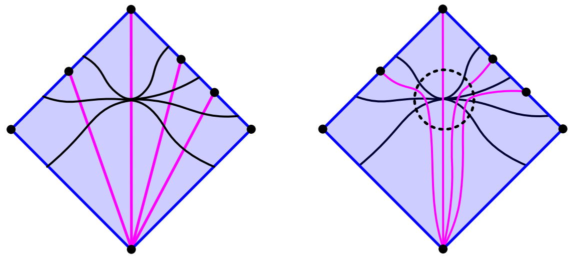

FIGURE 11. In the proof of property (5), we homotope edges of the flow graph as shown. The dashed circle indicates the intersection of a small neighborhood of a τ-edge with a sector.

Thus, if we now cut this modified c along all its intersections with L, we obtain sequences of simple $h_1(G_f)$ edges, interspersed with sequences of Φ_N edges that correspond to non-simple edges of $h_1(G_f)$. This decomposition gives us a cycle c' in G_f, whose h_2 image is c. Moreover, the number of edges in c' is equal to the number of points of $c \cap L$. This establishes point (5).

We can now give a proof of Claim 6.8. First, from [LMT21, Theorem 7.1] we know that

$$\lambda(f) = \mathrm{gr}_{\Phi_N}(\xi),$$

where $\xi \in H^1(N)$ is the cohomology class dual to L. Indeed, the graph Φ encodes the flow [LMT21, Theorem 6.1] and the directed cycles of $\Phi|\eta$ are in correspondence with the flow orbits that avoid Σ. Restricting to Φ_N gives us the flow orbits that cross L, so that ξ records size of the f-orbits associated to each of them.

Thus Claim 6.8 follows from the equality

$$\mathrm{gr}_{\Phi_N}(\xi) = \mathrm{gr}_{G_f}(1),$$

which is a consequence of item (5) above: it gives a bijection between cycles of G_f and Φ_N, so that the class that counts each edge of G_f exactly once is taken to ξ.

6.2.4. *Finishing the proof of Theorem 6.4.* Let $w_{ij}^{(n)}$ be the number of times the branch e_i of $\mathcal{V}$ maps over the branch e_j under $(f_{\mathcal{V}})^n$. This quantity also counts the number of directed paths of length n in G_f from the vertex e_i to the vertex e_j. Note that for each i and $n \geqslant 1$, $\sum_j w_{ij}^{(n)}$ is finite and similarly for each j and $n \geqslant 1$, $\sum_i w_{ij}^{(n)}$ is finite.

Our next claim is straightforward given the following observation: the endperiodicity of f implies that all but finitely many branches of $\mathcal{V}$ are mapped homeomorphically by $f_{\mathcal{V}}$ to other branches of $\mathcal{V}$. This implies that each end of G_f has a neighborhood homeomorphic to a ray.

Claim 6.9.

$$\mathrm{gr}_{G_f}(1) = \max_i \limsup_{n\to\infty} \left(\sum_j w_{ij}^{(n)}\right)^{1/n}. \tag{2}$$

Sketch. As observed above, there is a finite subgraph G' of G_f such that each complementary component of G' is a directed ray. Then both sides of Equation (2) are unchanged if G_f is replaced by G' (that is, when the right-hand side counts directed paths in G'). But the equality in Equation (2) is well-known for finite graphs, see e.g. [McM15, Lemma 3.1]. $\square$

All the pieces are now in place for the proof of Theorem 6.4 which topologically characterizes the stretch factor.

Proof of Theorem 6.4. Let α and β be any curves on L. Fix any representative b of the homotopy class of β on L that is transverse to the branches of $\mathcal{V}$ and let a be a representative of α's homotopy class contained in—not necessarily carried by—the track $\mathcal{V}$. That such a representative exists is consequence of the fact that the complementary regions of $\mathcal{V}$ are n-gons. Let a_n be the image of a after taking $f^n(a) \subset f^n(\mathcal{V})$ and folding it back into $\mathcal{V}$, and observe that $i(\beta, f^n(\alpha)) \leqslant \#(b \cap a_n)$.

After reindexing the branches of $\mathcal{V}$, we assume that a traverses the branches $c_1e_1, \ldots, c_ke_k$ where the c_i are the positive integer multiplicities. Let d_j be the number of intersections that b has with e_j. Then

$$\#(b \cap a_n) = \sum_j d_j \sum_{i=1}^k w_{ij}^{(n)} c_i.$$

Using the above together with Claim 6.9 and Claim 6.8 we conclude that

$$\begin{aligned}
\limsup_{n\to\infty} i(\beta, f^n(\alpha))^{1/n} &\leqslant \limsup_{n\to\infty} \#(b \cap a_n)^{1/n} \\
&= \limsup_{n\to\infty} \left(\sum_j d_j \sum_{i=1}^k w_{ij}^{(n)} c_i \right)^{1/n} \\
&\leqslant \max_i \limsup_{n\to\infty} \left(\sum_j w_{ij}^{(n)} \right)^{1/n} \\
&\overset{6.9}{=} \mathrm{gr}_{G_f}(1) \\
&\overset{6.8}{=} \lambda(f),
\end{aligned}$$

and hence

$$\max_{\alpha,\beta} \limsup_{n\to\infty} i(\beta, f^n(\alpha))^{1/n} \leqslant \lambda(f). \tag{3}$$

We now aim to prove the reverse of Equation (3). Let C be a recurrent component of G_f with growth rate $\lambda(f)$. This subgraph C exists because $\lambda(f) = \mathrm{gr}_{G_f}(1)$ by Claim 6.8, and because the growth rate of a directed graph is always achieved by a recurrent component.

We can find a curve α carried on $\mathcal{V}$ that traverses every branch of $\mathcal{V}$ associated to a vertex of C as follows. Let e be a branch of $\mathcal{V}$ corresponding to a vertex of C. Then the train paths $f^n(e)$ in $\mathcal{V}$ eventually cross every branch associated to C (and possibly other branches as well). By surgery at a $\mathcal{V}$-branch corresponding to a vertex of C crossed multiple times by some $f^n(e)$, we can find a closed curve carried by $\mathcal{V}$ that traverses this $\mathcal{V}$-branch. This curve may not be embedded, but we can resolve any points of intersection to obtain an embedded multicurve crossing the same set of branches. Applying f sufficiently many times to a given component of this multicurve yields a simple closed curve traversing each branch corresponding to C. Let α be this curve.

Next, note that since $f^n(\alpha)$ is not eventually contained in any compact set of L up to homotopy by Lemma 2.1, it eventually crosses a positive juncture β. From Claim 6.6 and Remark 6.7, we can realize β so that it is transverse to $\mathcal{V}$ and there are no complementary bigons. In particular, all of the carried curves $f^n(\alpha)$ are in minimal position with respect to β and hence their geometric intersection number is given by counting their points of intersection.

To complete the proof, reindex so that $e_1, \ldots, e_\ell$ are the vertices of C, which we identify with the corresponding branches of $\mathcal{V}$, each of which is traversed by α with positive weights $c_1, \ldots, c_\ell$. Also let $e_{\ell+1}, \ldots, e_m$ be the branches intersected by β, and d_i be the number of intersections of β with e_i for $\ell < i \leqslant m$. (Note that we do not assume that the (e_i) lie in C for $i > \ell$.) We will also use the symbol e_i to denote the corresponding vertex of G_f.

The fact that $f^n(\alpha)$ intersects β for some $n \geqslant 1$ translates into the fact that for some fixed $k > \ell$, there is a directed path p in G_f from a vertex of C to e_k. For this fixed k, each path of length n from the vertex e_i of C to e_k contributes $c_i d_k$ points of intersection to $i(\beta, f^n(\alpha))$. Then

$$i(\beta, f^n(\alpha)) \geqslant \sum_{i=1}^{\ell} w_{ik}^{(n)} c_i d_k,$$

and so it suffices to apply the fact that

$$\limsup_{n\to\infty} \left(\sum_i (A^n)_{ik} \right)^{\frac{1}{n}} \geqslant \lambda,$$

whenever C is a component with growth λ of a directed graph with transition matrix A, k is a fixed vertex of the graph reachable from a vertex of C, and the sum is over all vertices of C. As C was chosen so that $\lambda = \lambda(f)$, the proof is complete. $\square$

7. Foliation cones and entropy functions

7.1. **Foliation cones via fibered cones.** In [CC99, CC17], Cantwell and Conlon show that if N is a sutured manifold that admits a taut, depth one foliation suited to N, and each component of $\partial_\pm N$ has negative Euler characteristic, then there exists a finite collection of disjoint open, convex, polyhedral cones in $H^1(N)$, called **foliation cones**, so that the integer points of each cone are exactly the foliated classes. See also [CCF19, Section 13]. Here, we recall from Section 1.2 that a class in $H^1(N)$ is foliated if it is dual to a depth one foliation suited to N; by Theorem 2.2 each depth one foliation is determined by its cohomology class, up to isotopy.

In this section, we show that for the atoroidal sutured manifolds considered here, one can completely understand the foliation cones of N as pullbacks of fibered cones of a closed 3-manifold. We emphasize that this independently develops much of the Cantwell–Conlon foliation cone theory directly from the fibered face theory of Thurston [Thu86] and Fried [Fri79].

Theorem 7.1 (Foliation cones). *Let N be an atoroidal sutured manifold such that each component of $\partial_\pm N$ is a closed surface of genus at least 2. Then there is a closed hyperbolic 3-manifold M and an embedding $i\colon N \to M$ with the property that a class in $H^1(N)$ is foliated if and only if it is the pullback under $i^*\colon H^1(M) \to H^1(N)$ of a fibered class that is contained in a $i(\partial_+ N)$–adjacent fibered cone.*

Hence,

$$\{i^*(\mathcal{C}) : \mathcal{C} \text{ is a fibered cone of } M \text{ containing } [i(\partial_+ N)])\}$$

is exactly the set of foliation cones of $H^1(N)$.

Note that from this it is immediate that there are finitely many foliation cones and each one is polyhedral and rational since this is the case for fibered cones [Thu86]. Also, as in Remark 3.14, each depth one foliation suited to N is obtained from a fibration of M by spinning.

Toward the proof of Theorem 7.1, we construct the manifold M using the results from Section 3.2. Let $h\colon \partial_\pm N \to \partial_\pm N$ be a component-wise homeomorphism that acts by -1 on $H^1(\partial_\pm N)$ (or at least on the cone spanned by all juncture classes) and such that the h–double $M = M(N,h)$ is hyperbolic (see Lemma 3.12). As before, we identify N with its image in M and let $\mathcal{C}_1, \dots, \mathcal{C}_k$ be the fibered cones of M that contain $[\partial_+ N]$. (Note that $[\partial_+ N] = [\partial_- N]$ in $H^1(M)$.) Each cone $\mathcal{C}_i$ is associated to a pseudo-Anosov flow φ_i of M for which $\partial_\pm N$ is almost transverse. We replace each φ_i with its minimal dynamic blowup so that it is transverse to $\partial_\pm N$.

Proof. First observe that if ξ is a fiber class contained in $\mathcal{C}_i$, then it is represented by a fiber surface S that is a cross section of the flow φ_i. Just as in the proof of Theorem 3.3, S can be spun about $\partial_\pm N$ to produce a taut, depth one foliation $\mathcal{F}$ that is transverse to φ_i. Hence, the restriction of $\mathcal{F}$ to N represents the foliated class $i^*(\xi)$.

Conversely, let α be foliated class in $H^1(N)$ represented by a depth one foliation $\mathcal{F}$. For M fixed as above, Proposition 3.9 applies to the foliation $\mathcal{F}$ and therefore it along with Remark 3.14 gives that there is a φ_i–cross section S of M, whose fibered cone contains $[\partial_+ N]$ in its boundary, such that spinning S about $\partial_\pm N$ produces the foliation $\mathcal{F}$, up to isotopy. Hence, α is the pullback of the class dual to S in M.

We note that the above shows that there exists a taut, depth one foliation suited to N if and only if there exists a fibered cone of M containing $[\partial_+ N]$ in its boundary.

To complete the proof, it remains to show i^* takes each fibered cone onto a foliation cone. If not, since the images of fibered cones cover the foliation cones, there are two distinct fibered cones whose pullbacks overlap in $H^1(N)$. In this case, there is a foliation of N, and hence M, which is transverse to both φ_i and φ_j for $i \neq j$. But then we can de-spin this foliation as above (again see Remark 3.14) to produce a surface that is positively transverse to φ_i and φ_j and meets every orbit, a contradiction. $\square$

The following corollary characterizes integral classes of a foliation cone as those represented by foliations that are transverse to the restriction of an (honest) pseudo-Anosov flow.

Corollary 7.2. *The closed manifold M in Theorem 7.1 can be constructed so that the pseudo-Anosov flow φ associated to a fibered cone $\mathcal{C} \subset H^1(M)$ containing $[i(\partial_+ N)]$ is transverse to $i(\partial_\pm N)$. The integral points of the foliation cone $i^*(\mathcal{C})$ are exactly the points represented by depth one foliations that are transverse to the restricted semiflow φ_N.*

Proof. This was essentially established in the proof of Theorem 7.1, except that φ was the dynamic blow up of a pseudo-Anosov flow. However, by applying the proof of Theorem 4.2, we see that there is a choice of $h\colon \partial_\pm N \to \partial_\pm N$ so that the flow φ on $M = M(N,h)$ transverse to $\partial_\pm N$ is an honest pseudo-Anosov flow. In fact, this can be achieved by taking h on each component to be a hyperelliptic involution composed with a sufficiently high power of a fixed pseudo-Anosov homeomorphism that acts trivially on H_1 (see Lemma 3.12). Taking a larger power if necessary, we see that M can be chosen so that each φ_i is transverse to $\partial_\pm N$ for each of the (finitely many) fibered cones containing $[i(\partial_+ N)]$ in its boundary. $\square$

7.2. **Entropy functions of foliation cones.** Fix N as in the statement of Theorem 7.1 and let $\mathcal{C} \subset H^1(N)$ be one of its foliation cones. For each depth one foliation $\mathcal{F}$ suited to N, we let $h_{\mathcal{F}} \colon L \to L$ be its monodromy, defined up to isotopy, where L is a depth one leaf of $\mathcal{F}$. Let $\lambda([\mathcal{F}])$ denote the stretch factor of any spA representative of $h_{\mathcal{F}}$. By Corollary 6.2 and Theorem 2.2, this depends only on the class $[\mathcal{F}] \in H^1(N)$.

Theorem 7.3. *Let N be a sutured manifold such that each component of $\partial_\pm N$ is a closed surface of genus at least 2, and let $\mathcal{C} \subset H^1(N)$ be a foliation cone of N. Then the function*

$$[\mathcal{F}] \mapsto \log \lambda([\mathcal{F}])$$

that assigns to a depth one foliation the logarithm of the stretch factor of its monodromy extends to a convex, continuous function $\mathrm{ent} \colon \mathcal{C} \to [0, \infty)$.

Proof. Let M and φ be as in Corollary 7.2. Note that for any $\mathcal{F}$ with $\xi = [\mathcal{F}] \in \mathcal{C}$ with depth one leaf L, the first return map to L under φ_N is an spA map $f \colon L \to L$. Hence, $\lambda([\mathcal{F}]) = \lambda(f)$ is exactly $\mathrm{gr}_{\varphi_N}(\xi)$, the growth rate of closed orbits of φ_N with respect to class ξ. In the terminology of [LMT21], ξ is strongly positive since it is the pullback of a fibered class in M (Theorem 7.1). Hence, [LMT21, Theorems 8.1, 9.1] imply that $\mathrm{ent}_{\varphi_N}(\xi) = \log(\mathrm{gr}_{\varphi_N}(\xi))$ defines a continuous, convex function on $\xi \in \mathcal{C}$. ☐

7.3. **spA maps as limits of pseudo-Anosovs and accumulations of stretch factors.** Let M be a closed, hyperbolic fibered manifold and let $\mathcal{C} \subset H^1(M)$ be a fibered cone with associated pseudo-Anosov suspension flow φ.

Let α_i be a sequence of primitive integral classes in the interior of $\mathcal{C}$, each one representing a cross section S_i with first return map f_i; we set $\lambda(\alpha_i) = \lambda(f_i)$ to be the stretch factor of the pseudo-Anosov f_i.

In [LMT21, Theorem 9.10], we characterize the limit set of all stretch factors arising from a single fibered cone $\mathcal{C}$, answering a question of Chris Leininger. Here we show that all such limits are themselves stretch factors of spun pseudo-Anosov maps:

Theorem 7.4. *With the setup above, if $\lambda(\alpha_i) \to \lambda > 1$, then λ is the stretch factor of an spA map $f \colon L \to L$.*

Moreover, there is a face $\mathcal{C}' \subset \partial\mathcal{C}$ such that for any primitive integral η in the relative interior of $\mathcal{C}'$, L can be chosen to be a leaf of a depth one foliation of M that is obtained by spinning a fiber surface of M about a surface Σ that is dual to η and almost transverse to φ.

Note that if η and α are primitive integral classes in $\partial\mathcal{C}$ and $\mathrm{int}(\mathcal{C})$, respectively, then $\alpha_i = \alpha + i\eta$ for $i \in \mathbb{N}$ satisfy the above conditions so long as there are infinitely many closed primitive orbits of φ that are η–null (see [LMT21, Corollary 9.8 and Lemma 9.11]). Here, an orbit γ is η–null if $\eta(\gamma) = 0$.

Proof. For $\alpha \in \mathrm{int}(\mathcal{C})$ and $\eta \in \mathcal{C}$, let $\mathrm{gr}_\varphi(\alpha; \eta)$ be the growth rate of η–null closed orbits of φ with respect to α.

Claim 9.11 in [LMT21] implies that, after passing to a subsequence, there exists a subface $\mathcal{C}'$ in the boundary of $\mathcal{C}$ such that for any primitive integral η in the relative interior of $\mathcal{C}'$, the sequence $\mathrm{gr}_\varphi(\alpha_i; \eta)$ is eventually constant and that

$$\lambda(\alpha_i) = \mathrm{gr}_\varphi(\alpha_i) \to \mathrm{gr}_\varphi(\alpha; \eta),$$

where α is equal to some α_i for i sufficiently large. See, in particular, the argument that occurs immediately after Lemma 9.12 in [LMT21]. We remark that the proof [LMT21, Claim 9.11] is only explicitly given for the manifold obtained from M by removing the singular orbits of φ, but essentially the same proof applies here.

Now let Σ be a surface dual to η that is almost transverse to φ (Theorem 2.3) and let $\varphi^\sharp$ be a dynamic blowup of φ that is transverse to Σ. Also let S be a cross section of φ (and hence $\varphi^\sharp$) that is dual to α, and let $\mathcal{F}$ be the $\varphi^\sharp$–transverse foliation of M obtained by spinning S about Σ. If we cut M along Σ, we obtain a disjoint union $N_1, \dots, N_k$ of manifolds with induced foliations $\mathcal{F}_1, \dots, \mathcal{F}_k$ transverse to semiflows $\varphi_{N_1}, \dots, \varphi_{N_k}$.

Since the η–null orbits of $\varphi^\sharp$ are exactly the orbits missing Σ, and hence contained in some N_i, we have

$$\begin{aligned} \mathrm{gr}_\varphi(\alpha;\eta) &= \mathrm{gr}_{\varphi^\sharp}(\alpha;\eta) \\ &= \max_i \mathrm{gr}_{\varphi_{N_i}}(\alpha) \\ &= \max_i \lambda(f_i), \end{aligned}$$

where f_i is the spA return map to a depth one leaf L_i of $\mathcal{F}_i$. This completes the proof. $\square$

8. Connection to Handel–Miller theory

In this section, we relate the structure of spun pseudo-Anosov maps developed here to Handel–Miller theory.

8.1. **Handel–Miller basics.** Let $g\colon L \to L$ be an endperiodic map. Fix a standard hyperbolic metric on L, and choose systems of g-junctures (see Section 4.3) for all g-cycles of ends of L. In [CCF19] it is proven that the geodesic tightenings of the negative g-junctures limit on a geodesic lamination Λ^+_{HM} and symmetrically the geodesic representatives of the positive g-junctures limit on a geodesic lamination Λ^-_{HM}. The laminations Λ^+_{HM} and Λ^-_{HM} are called the positive and negative **Handel–Miller laminations**, respectively. These laminations are mutually transverse. Furthermore by [CCF19, Corollary 4.72] $\Lambda^\pm_{\mathrm{HM}}$ are independent of the choice of junctures, and by [CCF19, Corollary 10.16], $\Lambda^\pm_{\mathrm{HM}}$ are independent of the choice of standard hyperbolic metric up to an ambient isotopy of L.

One property of $\Lambda^\pm_{\mathrm{HM}}$ we will use below is that each leaf of say Λ^+_{HM} accumulates on a positive end and hence intersects positive g-junctures.

By [CCF19, Theorem 4.54], g is isotopic to an endperiodic map $h\colon L \to L$ which permutes the set of tightened g-junctures and leaves invariant $\Lambda^\pm_{\mathrm{HM}}$. This map h is called a **Handel–Miller map**, or sometimes a **Handel–Miller representative** of the isotopy class of g. While h is not uniquely determined, it is uniquely determined on $\Lambda^+_{\mathrm{HM}} \cap \Lambda^-_{\mathrm{HM}} \subset L$.

Let $\mathfrak{U}_+$ be the set consisting of all points in L that escape to the positive ends of L under iteration of h, and $\mathfrak{U}_-$ the set consisting of points that escape to the negative ends of L under iteration of h^{-1}. We call $\mathfrak{U}_+$ and $\mathfrak{U}_-$ the positive and negative **escaping sets**, respectively. By [CCF19, Lemma 4.71], $\Lambda^+_{\mathrm{HM}} = \partial\mathfrak{U}_-$ and $\Lambda^-_{\mathrm{HM}} = \partial\mathfrak{U}_+$.

Let $\mathcal{P}_+ = L - (\Lambda^+_{\mathrm{HM}} \cup \mathfrak{U}_-)$, and symmetrically $\mathcal{P}_- = L - (\Lambda^-_{\mathrm{HM}} \cup \mathfrak{U}_+)$. A component of $\mathcal{P}_+$ or $\mathcal{P}_-$ is called a positive or negative **principal region**, respectively. A negative principal region is a positive principal region of h^{-1} and vice versa.

The structure of principal regions is developed in detail in [CCF19, Sections 5.3, 6.1-6.4]. In the case of interest to us, when L is without boundary and h is atoroidal, this is particularly simple and directly analogous to what happens for pseudo-Anosov maps of closed surfaces. We will sketch the structure with these hypotheses, which we will be assuming from now on.

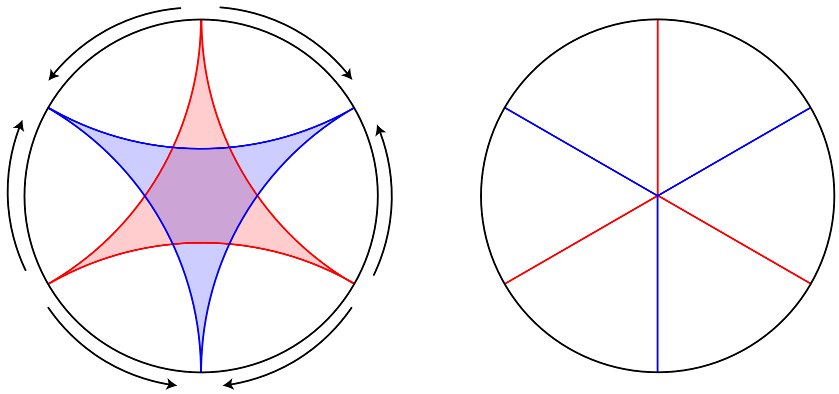

FIGURE 12. Left: lifts of two dual principal regions to $\widetilde{L}$, as well as the boundary dynamics of the action of a lift of a power of f fixing the ends of two lifted principal regions. Right: the corresponding lifts of singular leaves of $\Lambda^{\pm}$.

Each positive principal region P_+ is homeomorphic to an open disk, and any lift $\widetilde{P}_+$ to the universal cover $\widetilde{L}$ is the interior of a finite-sided polygon with vertices in $\partial\widetilde{L}$. This lift has a *dual* lift $\widetilde{P}_-$ of a negative principal region, whose vertices on $\partial\widetilde{L}$ alternate with those of $\widetilde{P}_+$ (see Figure 12). A suitable power of a lift of h preserves the vertices of both dual polygons. The dynamics of this lift on the circle at infinity are as follows:

Lemma 8.1. *Let $h\colon L \to L$ be an Handel-Miller atoroidal endperiodic map. Let $\widetilde{P}_+$ be the lift of a positive principal region of h with dual lifted negative principal region $\widetilde{P}_-$. Suppose $\widetilde{h^p}$ is a lift of a power of h to $\widetilde{L}$ that preserves P_+, and hence P_-, as well as all their vertices. Then the action of $\widetilde{h^p}$ on $\partial\widetilde{L}$ is multi sink-source, where the sources are exactly the vertices of $\widetilde{P}_-$ and the sinks are the vertices of $\widetilde{P}_+$.*

Lemma 8.1 follows from Proposition 6.9(ii) and Corollary 6.13 [CCF19] since in our setting lifts of principal regions have finitely many ideal vertices on the circle at infinity.

8.2. **Spun pseudo-Anosov maps are Handel–Miller.** We collect the remaining needed properties of the Handel–Miller map in this proposition:

Proposition 8.2. *Let $h\colon L \to L$ be a Handel–Miller map with invariant laminations $\Lambda^{\pm}_{\mathrm{HM}}$. Then the collection of periodic leaves of $\Lambda^{\pm}_{\mathrm{HM}}$ is dense in $\Lambda^{\pm}_{\mathrm{HM}}$.*

If $\widetilde{h}\colon \widetilde{L} \to \widetilde{L}$ is any lift of a power of h that fixes a point $\widetilde{q}$ of $\widetilde{L}$, then either $\widetilde{q}$ is contained in the closure of a lift of a principal region or the following holds: $\widetilde{q}$ is the unique fixed point of $\widetilde{h}$ in $\widetilde{L}$, it is the point of intersection between two leaves $\ell^{\pm} \subset \widetilde{\Lambda}^{\pm}_{\mathrm{HM}}$, and $\partial\ell^{\pm} \subset \partial\widetilde{L}$ are exactly the fixed points of $\widetilde{h}^p$ on $\partial\widetilde{L}$, where p is the smallest natural number such that $\widetilde{h}^p$ fixes the ends of $\ell^{\pm}$.

Proof. By [CCF19, Proposition 5.12], the leaves of $\Lambda^{\pm}_{\mathrm{HM}}$ which border $\mathcal{U}_{\mp}$ are dense in $\Lambda^{\pm}_{\mathrm{HM}}$. By [CCF19, Theorem 6.5], each such leaf is periodic. This proves the first claim.

For the second claim, note that the projection of $\widetilde{q}$ cannot lie, in particular, in the negative escaping set. By the definition of principal regions we have $L = \mathcal{P}_+ \cup \mathcal{U}_- \cup \Lambda^{+}_{\mathrm{HM}}$, so we conclude that $\widetilde{q}$ must lie in the closure of a lift of a principal region or in the lift of a leaf of Λ^+_{HM}. We remark that the two cases are not mutually exclusive since positive principal regions are bordered by leaves of Λ^+_{HM}.

Thus it suffices to consider only the case when $\widetilde{q}$ lies in a lifted leaf ℓ^+ of $\widetilde{\Lambda}^+_{\mathrm{HM}}$ which does *not* lie on the boundary of a lifted principal region.

Let p be the least natural number such that $\widetilde{h}^p$ fixes both ends of ℓ^+. Let $\widetilde{\sigma}$ be the lift of a positive h-juncture in L intersecting ℓ^+. Since h is Handel–Miller and $\widetilde{h}^p$ fixes the ends of ℓ^+, under iteration of $\widetilde{h}^{-p}$ we have that $\widetilde{\sigma}$ must converge to a leaf ℓ^- of $\widetilde{\Lambda}^-_{\mathrm{HM}}$ from one of its sides. Under iteration of $\widetilde{h}^p$, $\widetilde{\sigma}$ must converge to a point in $\partial\widetilde{L}$ since its image in L escapes to the positive ends of L under iteration of h. In fact, the sequence converges to the boundary point of ℓ^+ that lies to the same side of ℓ^+ intersecting $\widetilde{\sigma}$.

Let $\widetilde{\sigma}'$ be another lifted h-juncture intersecting ℓ^+ on the opposite side of ℓ^- as $\widetilde{\sigma}$. Then under iteration of $\widetilde{h}^{-p}$, $\widetilde{\sigma}'$ converges to a leaf ℓ^-_1 of $\widetilde{\Lambda}^-_{\mathrm{HM}}$. By [CCF19, Corollary 6.15], we must have that $\ell^- = \ell^-_1$ because otherwise ℓ^+ would bound the lift of a principal region. This forces $\widetilde{q}$ to be the unique point in $\ell^+ \cap \ell^-$.

Furthermore, the picture of lifted h-junctures developed above shows that every point in $\widetilde{L}$ limits under iteration of $\widetilde{h}^{-p}$ to $\widetilde{q}$ or a boundary point of ℓ^- in ∂L. It follows that $\widetilde{q}$ is the unique fixed point of $\widetilde{h}^p$. The statement about dynamics on $\partial\widetilde{L}$ also follows. ☐

Remark 8.3. Proposition 8.2 is true without the assumption that h is atoroidal, but in that case the lifts of principal regions are more complicated than the picture developed in Section 8.1.

Recall from Section 5.2, that if Λ is a (possibly singular) lamination on L, then $\partial^2(\Lambda)$ denotes the closed, pairwise unlinked, $\pi_1(L)$–invariant subset of $\partial^2\widetilde{L}$ obtained by taking the boundaries of *leaf lines* in $\widetilde{\Lambda} \subset \widetilde{L}$. Note that if the lamination is nonsingular, then every leaf is regular and hence a leaf line.

Theorem 8.4. *Suppose that $f\colon L \to L$ is an spA map and h is a homotopic Handel–Miller map. Then as $\pi_1(L)$–invariant subspaces of $\partial^2\widetilde{L}$,*

$$\partial^2(\Lambda^\pm) = \partial^2(\Lambda^\pm_{\mathrm{HM}}).$$

Proof. Combining Proposition 8.2 and Theorem 5.3 it suffices to prove that $\partial^2(\Lambda^\pm)$ and $\partial^2(\Lambda^\pm_{\mathrm{HM}})$ have the same set of periodic points (i.e. points fixed under a lift of some power of the homotopy class of f).

Let ℓ be a periodic leaf line of either $\widetilde{\Lambda}^+$ or $\widetilde{\Lambda}^-$. By Remark 5.9, there is an $n \geqslant 1$, a lift $\widetilde{f}^n$ of f^n, and a point $\widetilde{p} \in \ell$ such that $\widetilde{f}^n$ fixes p and preserves ℓ. By Proposition 4.11, $\widetilde{f}^n$ has multi sink-source dynamics on $\partial\widetilde{L}$. Let $\widetilde{h}$ be the corresponding lift of h so that $\widetilde{h}$ and $\widetilde{f}$ have the same action on $\partial\widetilde{L}$. Corollary 4.10 and Lemma 4.14 then imply that $\widetilde{h}^n$ fixes a point $\widetilde{q}$ in $\widetilde{L}$. According to Proposition 8.2, either $\widetilde{q}$ is the unique such point and we see that the end points of ℓ must agree with those of the leaf of $\Lambda^\pm_{\mathrm{HM}}$ through $\widetilde{q}$, or $\widetilde{q}$ is in the closure of the lift of a principal region. In the latter case, Lemma 8.1 shows that the endpoints of ℓ are shared by a leaf of $\widetilde{\Lambda}^\pm_{\mathrm{HM}}$ which is fixed under $\widetilde{h}^n$. This shows that $\partial^2(\Lambda^\pm) \subset \partial^2(\Lambda^\pm_{\mathrm{HM}})$.

If, instead, ℓ is a periodic leaf of (say) $\widetilde{\Lambda}^+_{\mathrm{HM}}$, then Proposition 8.2 furnishes a periodic point $\widetilde{q}$ contained in ℓ and also contained in a periodic leaf ℓ^- of $\widetilde{\Lambda}^-_{\mathrm{HM}}$, so that $\ell \cap \ell^- = \widetilde{q}$. Indeed, if ℓ does not bound a lifted principal region then there is nothing to say, and if ℓ does bound a lifted principal region then there are two possible choices of $\widetilde{q}$ and ℓ^-.

Let $\widetilde{h}$ be a lift of h and $n \geqslant 1$ so that $\widetilde{h}^n$ fixes both ℓ and ℓ^-. Then Lemma 4.9 implies that the endpoints of ℓ are sinks and the endpoints of ℓ^- are sources, both for the action of $\widetilde{h}^n$ on $\mathbb{D}$ and of $\widetilde{f}^n$ on $\mathbb{D}$. So we may again apply Lemma 4.14 to conclude that $\widetilde{f}^n$ has a fixed point $\widetilde{p}$, where $\widetilde{f}$ is the $\widetilde{h}$–compatible lift of f. Note that $\widetilde{f}^n$ must fix the half-leaves of $\widetilde{\Lambda}^{\pm}$ from $\widetilde{p}$ because $\widetilde{f}^n$ has a nonempty fixed point set in $\partial\widetilde{L}$. Hence by Proposition 4.11, there exist leaves of $\widetilde{\Lambda}^+$ and $\widetilde{\Lambda}^-$ whose endpoints are, respectively, the sinks and sources of the action of $\widetilde{f}^n$. This shows that $\partial^2(\Lambda^{\pm}_{\mathrm{HM}}) \subset \partial^2(\Lambda^{\pm})$. □

Theorem 8.4 gives the precise relation between spA and Handel–Miller representatives. In particular, a spun pseudo-Anosov map $f\colon L \to L$ preserves the pair of singular laminations $\Lambda^{\pm}$, which according to Theorem 8.4 are singular versions of $\Lambda^{\pm}_{\mathrm{HM}}$ obtained by pinching principal regions to singular leaves (see Figure 12). It is in this precise sense that one can consider spA maps as 'singular' Handel–Miller maps.

Remark 8.5. Handel–Miller theory applies more generally to endperiodic maps $f\colon L \to L$ for which $\partial L \neq \varnothing$, as long as each noncompact component of ∂L joins a positive end to a negative end and contains no periodic points; see [CCF19, Hypothesis 3]. It would be interesting to generalize the theory of spA maps to cover this case as well. Potential difficulties include that (1) Proposition 4.6, which uses work of Fenley [Fen09], would need to be extended to the case with boundary, and (2) to extend Theorem 7.1 one needs a homeomorphism of a surface with boundary that sends every juncture class to its negative.

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